Homework 5 Solutions

- Section 9 (p.45) # 2, 6,
- Section 10 (p.54) # 2,6.

Section 9, Problem 2. Prove that $\Phi^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathcal{W}$; show also that

$$\frac{2(\beta^\vee,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)} = \frac{2(\alpha,\beta)}{(\beta,\beta)} \tag{1}$$

and draw a picture of $\Phi^\vee$ in the cases $\mathtt{A}_2$, $\mathtt{B}_2$, $\mathtt{G}_2$.

Solution. Recall that $\alpha^\vee$ is defined for nonzero α in a Euclidean space E as

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}.$$

Thus the vector $\alpha^\vee$ is proportional to α, but its length is inverted, so long vectors become short, and short vectors become long.

Lemma 1. *If we do this operation twice, we recover α. That is, $(\alpha^\vee)^\vee = \alpha$.*

Proof. Indeed,

$$(\alpha^\vee)^\vee = \frac{2\alpha^\vee}{(\alpha^\vee,\alpha^\vee)} = \frac{2\times 2\alpha/(\alpha\,\alpha)}{4(\alpha\,\alpha)/(\alpha\,\alpha)^2} = \alpha. \tag{2}$$

□

Now let us prove (1).

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \left(\alpha, \frac{2\beta}{(\beta,\beta)}\right) = (\alpha,\beta^\vee),$$

while using Lemma 1,

$$\frac{2(\beta^\vee,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)} = (\beta^\vee,(\alpha^\vee)^\vee) = (\beta^\vee,\alpha).$$

Comparing the last two identities proves (1).

I will use r_α to denote the reflection in the hyperplane orthogonal to a vector α. Humphreys uses the notation σ_α for r_α.

Lemma 2. *The transformation r_α is an isometry, that is*

$$(r_\alpha(x), r_\alpha(y)) = (x,y). \tag{3}$$

Proof. This is obvious from the characterization of the map as a reflection in a hyperplane through the origin. □

Lemma 3. *The reflections r_α and $r_{\alpha^\vee}$ are the same transformation of E.*

Proof. This is because r_α is the hyperplane orthogonal to α, but this is also the hyperplane orthogonal to the proportional vector $\alpha^\vee$, so the reflections are the same. $\square$

Lemma 4. *We have*

$$r_\alpha(\beta^\vee) = r_\alpha(\beta)^\vee.$$

Proof. This is because the vectors β and $r_\alpha(\beta)$ have the same length, by Lemma 2, so

$$r_\alpha(\beta)^\vee = \frac{2r_\alpha(\beta)}{(r_\alpha(\beta), r_\alpha(\beta))} = \frac{2r_\alpha(\beta)}{(\beta,\beta)} = r_\alpha\left(\frac{2\beta}{(\beta,\beta)}\right) = r_\alpha(\beta^\vee).$$

$\square$

Theorem 5. $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ *is a root system.*

Proof. If $\alpha^\vee \in \Phi^\vee$ then we need to show that $r_{\alpha^\vee}(\beta^\vee) \in \Phi^\vee$. This follows from the last two Lemmas since $r_{\alpha^\vee}(\beta^\vee) = r_\alpha(\beta^\vee) = r_\alpha(\beta)^\vee$, and $r_\alpha(\beta) \in \Phi$. We also need to know that if $\alpha^\vee, \beta^\vee \in \Phi^\vee$, then $2(\alpha^\vee, \beta^\vee)/(\alpha^\vee, \beta^\vee) \in \mathbb{Z}$, but this follows from (1). $\square$

Section 9, Problem 6. Prove that $\mathcal{W}$ is a normal subgroup of $\mathrm{Aut}(\Phi)$, the group of all linear isomorphisms of Φ onto itself.

Solution. As in the last notation, I am using r_α for the reflection in a root or more generally a nonzero vector in the ambient space E, which in Humphreys' notation is σ_α.

We recall the formula:

$$\sigma r_\alpha \sigma^{-1} = r_{\sigma(\alpha)} \tag{4}$$

from the Lemma in Section 9.2 (p.43). Humphreys asserts this for $w \in W$, but it is actually true if σ is an isometry, that is, $(\sigma(x), \sigma(y)) = (x, y)$ for $x, y \in E$. According to the definition (page 43) automorphisms of the root system are assumed to be isometries.

To prove (4) if σ is an isometry, we can either note that since σ, r_α and σ are all isometries, the left-hand side is an isometry that fixes the hyperplane orthogonal to $\sigma(\alpha)$, and maps $\sigma(\alpha)$ to its negative, hence agrees with $r_{\sigma(\alpha)}$. Alternatively, we can just compute:

$$\sigma r_\alpha \sigma^{-1}(x) = \sigma\left(\sigma^{-1}(x) - \frac{2(\alpha, \sigma^{-1}x)}{(\alpha,\alpha)}\alpha\right) = \sigma\left(\sigma^{-1}(x) - \frac{2(\sigma(\alpha), \sigma\sigma^{-1}(x))}{(\sigma(\alpha), \sigma(\alpha))}\alpha\right)$$

$$= x - \frac{2(w(\alpha), x)}{(w(\alpha), w(\alpha))} w(\alpha) = r_{w(\alpha)}(x).$$

Now we can prove that W is normal in $\mathrm{Aut}(\Phi)$. If σ is an automorphism of Φ, then since W is generated by the r_α with $\alpha \in \Phi$, it is sufficient to show that $\sigma r_\alpha \sigma^{-1} \in W$, but this is an immediate consequence of (4).

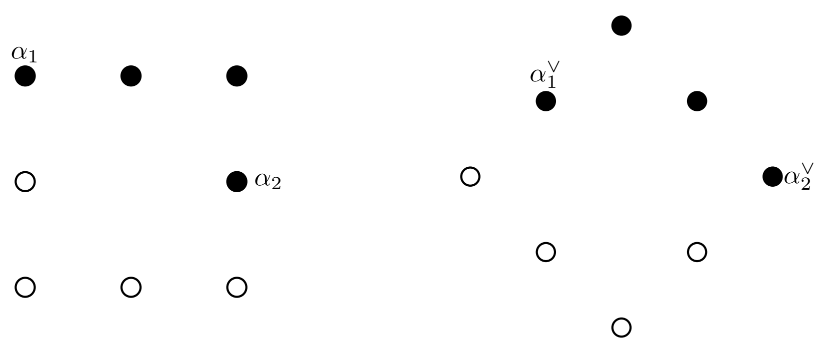
Section 10, Problem 2. If Δ is a base of Φ, and $\alpha, \beta \in \Delta$ ($\alpha \neq \beta$), prove that the set $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$ is a root system of rank 2 in the subspace E spanned by α, β (see Exercise 9.7).

Solution. This is a very useful observation, since the rank two root systems are easily classified as $A_1 \times A_1$, A_2, $B_2 = C_2$ or G_2.

Let $V \subset E$ be the vector subspace spanned by α, β. We claim that $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi = V \cap \Phi$. Indeed, if $\gamma \in V \cap \Phi$ we may write $\gamma = \sum_{\delta \in \Phi^+} n_\delta \cdot \delta$ where $n_\delta \in \mathbb{Z}$. Because the elements of Δ are linearly independent, we must have $n_\delta = 0$ unless $\delta = \alpha$ or β, so $\gamma \in (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$.

With this in mind, the axioms of a root system for $V \cap \Phi$ are easily verified. If $\gamma \in V \cap \Phi$ then $r_\gamma(V) = V$ follows from the formula $r_\gamma(x) = x - (x, \gamma^\vee)\gamma$, and $r_\gamma(\Phi) = \Phi$ since Φ is a root system, so $r_\gamma(V \cap \Phi) = V \cap \Phi$. The Cartan numbers $2(\gamma, \delta)/(\gamma, \gamma)$ are integers for $\gamma, \delta \in V \cap \Phi$ since Φ is a root system.

We are also asked to draw some pictures. I'll just do $\mathtt{B}_2$ and its dual root system, which is isomorphic to $\mathtt{C}_2$. Note how if α is a long root, then $\alpha^\vee$ is a short root.



Section 10, Problem 6. Define a function $\mathrm{sn} : \mathcal{W} \longrightarrow \{\pm 1\}$ by $\mathrm{sn}(\sigma) = (-1)^{\ell(\sigma)}$. Prove that sn is a homomorphism.

Solution. Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$, and let $s_i = r_{\alpha_i}$ be the corresponding simple reflections.

Lemma 6. *As a linear transformation of E,* $\det(s_i) = -1$.

Proof. Choose a basis $v_1, \cdots, v_\ell$ of E such that $v_1 = \alpha_i$ is the first basis vector, and $v_2, \cdots, v_\ell$ are orthogonal to v_1. Then $v_1, \cdots, v_\ell$ are eigenvectors of s_i, with eigenvalues $-1, 1, \cdots, 1$. So the determinant of s_i is the product of the eigenvalues, or -1. □

Now if $w \in W$ write $w = s_{i_1} \cdots s_{i_k}$ where $k = \ell(w)$. Then $\mathrm{sn}(w) = (-1)^k = \det(w)$. Since the sign of w is thus its determinant, it is multiplicative.

Note: An alternative solution may be based on the exchange principle (Lemma C on page 50).