Math 210C Homework 4

- Section 7 (p.34) # 2,6,7,
- Section 8 (p.40) # 5,8,11.

Section 7 #2. Let $M = \mathfrak{sl}(3,F)$. Then M contains a copy of $L = \mathfrak{sl}(2,F)$ in its upper left-hand corner. Write M as a direct sum of irreducible L-submodules (M viewed as an L-module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Solution: We will use this notation for $\mathfrak{sl}(2)$:

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These are denoted h, x, y by Humphreys.

Here is a more general strategy for decomposing a module into $\mathfrak{sl}(2)$ irreducibles. We recall from Section 7 that the irreducible module $V(m)$ contains a *highest weight vector* v_0 which is characterized up to constant multiple by the condition that $Ev_0 = 0$. If v_0 is found, then m can be recovered because $Hv_0 = \lambda v_0$. Therefore we arrive at the following way of decomposing an $\mathfrak{sl}(2)$ module W into irreducibles. If

$$W = V(m_1) \oplus V(m_2) \oplus \cdots \oplus V(m_N),$$

let

$$W_0 = \{v \in W | Ev = 0\}.$$

Then W_0 is spanned by the highest weight vectors, one for each component $V(m_i)$. Moreover, since the highest weight vector in $V(m_i)$ is an eigenvector of H with eigenvalue m_i, the multiplicity of the eigenvalue λ of H in W_0 is the number of m_i that are equal to λ.

Now embedding $\mathfrak{sl}(2) \longrightarrow \mathfrak{sl}(3)$ in the upper-left corner, we are asked to decompose $\mathfrak{sl}(3)$, which is then an $\mathfrak{sl}(2)$ module into irreducibles. Applying the technique just explained with $W = \mathfrak{sl}(3)$ the space

$$W_0 = \{X \in \mathfrak{sl}(3) | \operatorname{ad}(E)X = 0\}, \qquad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Writing

$$X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix}$$

the condition to be in W_0 is that $\operatorname{ad}(E)X = 0$, where

$$\operatorname{ad}(E_{12})X = EX - XE = \begin{pmatrix} d & e-a & f \\ 0 & -d & 0 \\ 0 & -g & 0 \end{pmatrix}.$$

Thus W_0 is characterized by $d = f = g = 0$ and $a = e$:

$$W_0 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & h & -2a \end{pmatrix} \right\}.$$

Now, as we explained, the multiplicity of $V(m)$ in the decomposition into irreducibles is the multiplicity of m as an eigenvalue of $\mathrm{ad}(H)$, where under the embedding $\mathfrak{sl}(2) \longrightarrow \mathfrak{sl}(3)$

$$H = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}.$$

We compute

$$\mathrm{ad}(H) \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & h & -2a \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a & -b & 0 \\ 0 & -a & 0 \\ 0 & -h & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b & c \\ 0 & 0 & 0 \\ 0 & h & 0 \end{pmatrix}.$$

From this we can see the eigenvalues:

Eigenvector $X \in W_0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
Eigenvalue of $\mathrm{ad}(H)$	0	2	1	1

Thus we arrive at the decomposition

$$\mathfrak{sl}(3) = V(0) \oplus V(1) \oplus V(1) \oplus V(2).$$

As a check, the dimensions are $\dim(\mathfrak{sl}(3)) = 8$, and on the right-hand side, since $\dim\ V(m) = m + 1$, the dimension is $1 + 3 + 2 + 2 = 8$.

The modules can be made explicit. The $V(2)$ is just $\mathfrak{sl}(2)$ embedded in the left corner, the $V(0)$ is just the linear span of its basis vector, and the 2-dimensional $V(1)$ modules are:

$$\left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix} \right\}.$$

Section 7 #6. Decompose the tensor product of the $L = \mathfrak{sl}(2)$ modules $V(3)$, $V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

Solution. We will give a different general method of decomposing an $\mathfrak{sl}(2)$ module into irreducibles than we used in the previous example. If W is an $\mathfrak{sl}(2)$ module, we may decompose this into H-eigenspaces:

$$W = \bigoplus_m W_m, \qquad W_m = \{v \in W | Hv = mv\}.$$

We only care about the dimensions of these, and we encode this information in a polynomial, the *character* χ_M:

$$\chi_M = \sum_m \dim(W_m) q^m$$

where q is an indeterminate. For example, by Humphreys Theorem 7.2, if W is the irreducible $V(m)$ then it is spanned by $v_0, \cdots, v_m$ where v_i spans a one-dimensional W_{m-2i}, so

$$\begin{gathered}
\chi_{V(0)} = 1 \\
\chi_{V(1)} = q + q^{-1} \\
\chi_{V(2)} = q^2 + 1 + q^{-2} \\
\vdots \\
\chi_{V(m)} = q^m + q^{m-2} + \ldots + q^{-m}.
\end{gathered}$$

Lemma 1. *If we can find integers $m_1, \cdots, m_k$ such that*

$$\chi_M = \chi_{V(m_1)} + \chi_{V(m_2)} + \ldots + \chi_{V(m_k)} \tag{1}$$

then

$$M \cong V(m_1) \oplus \cdots \oplus V(m_k).$$

Proof. There always exists a decomposition (1) by Weyl's theorem (Section 6.3) and the classification of irreducibles for $\mathfrak{sl}(2)$ (Section 7.2). The numbers m_i are determined by the character, because the characters of the irreducibles, $\chi_{V(m)} = q^m + q^{m-2} + \ldots + q^{-m}$ are easily seen to be linearly independent. □

Lemma 2. *The character is multiplicative, that is*

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

Proof. Let $x \in V_m$ and $y \in W_n$. Then we claim that $x \otimes y \in (V \otimes W)_{n+m}$. Indeed

$$H(x \otimes y) = Hx \otimes y + x \otimes Hy = mx \otimes y + x \otimes ny = (n+m)(x \otimes m).$$

From this observation

$$\sum_{m+n=N} V_m \otimes W_n = (V \otimes W)_N.$$

Thus

$$\begin{aligned}
\chi_{V \otimes W} &= \sum_N \dim(V \otimes W)_N \; q^N = \sum_N \sum_{m+n=N} \dim(V_n) \dim(W_m) q^N \\
&= \left(\sum_n \dim(V_n) q^n \right) \left(\sum_m \dim(W_m) q^m \right) = \chi_V \chi_W.
\end{aligned}$$

□

Now

$$\chi_{V(3)\otimes V(67)} = \chi_{V(3)} \cdot \chi_{V(7)} = (q^3 + q + q^{-1} + q^{-3})(q^7 + q^5 + q^3 + q^1 + q^{-1} + q^{-3} + q^{-5} + q^{-7})$$
$$= q^{10} + 2q^8 + 3\chi^6 + 4q^4 + 4q^2 + 4 + 4q^{-2} + 4q^{-3} + 3q^{-5} + 2q^{-7} + q^{-9}.$$

Our goal is to express this as a sum of characters of $V(m)$ for various m. The polynomials $\chi_{V(m)}$ are linearly independent so there is a unique way to do this. The complete decomposition is

$$\chi_{V(3)\otimes V(7)} = \chi_{V(10)} + \chi_{V(8)} + \chi_{V(6)} + \chi_{V(4)}.$$

Therefore

$$V(3) \otimes V(7) \cong V(10) \oplus V(8) \oplus V(6) \oplus V(4)$$

In general, to compute $V(m) \otimes V(n)$ we may assume that $m > n$. Then the answer is

$$V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n)$$

In the next exercise I am changing notation and writing $M(\lambda)$ instead of $Z(\lambda)$. The notation has become standardized in the years since Humphrey's book was written, and $M(\lambda)$ is nowadays called a *Verma module*. Humphreys calls $M(\lambda)$ a *standard cyclic module* and denotes it $Z(\lambda)$. Verma modules become important later in the book, where following BGG he uses these infinite dimensional modules to study the finite-dimensional irreducibles. I am not asking you to do (c), though (c) is not hard if you do the last part of (a).

Section 7 #7. In this exercise we construct certain *infinite-dimensional* L-modules. Let $\lambda \in F$ be an arbitrary scalar. Let $M(\lambda)$ be a vector space over F with countable basis $(v_0, v_1, v_2, \cdots)$.

(a) Prove that formulas (a)-(c) of Lemma (7.2) define an L-module structure on $M(\lambda)$, and that every nonzero L-submodule of $M(\lambda)$ contains at least one maximal vector.

(b) Suppose $\lambda + 1 = i$ is a positive integer. Prove that this induces an L-module homomorphism $M(\mu) \xrightarrow{\phi} M(\lambda)$, $\mu = \lambda - 2i$, sending v_0 in $M(\mu)$ to v_i in $M(\lambda)$.

Solution. The action of x, y, h on $M(\lambda)$ is as follows. $h(v_i) = (\lambda - 2i)v_i$, $x(v_0) = 0$ while

$$x(v_i) = (\lambda - i + 1)v_{i-1} \text{ if } i \geqslant 1, \qquad y(v_i) = (i+1)v_{i+1}. \tag{2}$$

To check that this is an L-module (where $L = \mathfrak{sl}_2$) we have to check three identities

$$hv_i = xyv_i - yxv_i, \qquad 2xv_i = hxv_i - xhv_i, \qquad -2yv_i = hyv_i - yhv_i \tag{3}$$

corresponding to the brackets $h = [x, y]$, $2x = [h, x]$ and $-2y = [h, y]$. We will only check the first one. If $i = 0$ then $yxv_0 = 0$ while $xyv_i = xv_1 = \lambda v_0$ so

$$hv_0 = \lambda v_0 = xyv_0 - yxv_0.$$

On the other hand if $i > 0$ then

$$xyv_i = (i+1)xv_{i+1} = (i+1)(\lambda - i)v_i, \qquad yxv_i = (\lambda - i + 1)yv_{i-1} = (\lambda - i + 1)iv_i.$$

Subtracting these equations

$$xyv_i - yxv_i = ((i+1)(\lambda - i) - (\lambda - i + 1)i)v_i = (\lambda - 2i)v_i = hv_i.$$

This is one of the identities in (3). For the next

$$hxv_i - xhv_i = h(\lambda - i + 1)v_{i-1} - x(\lambda - 2i)v_i = (\lambda - 2i + 2)(\lambda - i + 1)v_{i-1} - (\lambda - i + 1)(\lambda - 2i)v_{i-1}$$

$$= 2(\lambda - i + 1)v_{i-1} = 2xv_i.$$

We leave the list identity in (3) to the reader. This establishes that (3) gives a valid L-module structure on the vector space with basis $v_0, v_1, \cdots$.

You are also asked to show that every nonzero submodule of $M(\lambda)$ contains at least one maximal vector. Call the submodule in question V. By "maximal vector" Humphreys means a vector $v \in V$ such that $xv = 0$. If $\mu \in F$ let V_μ be the H-eigenspace $\{v \in V | Hv = \mu v\}$. If v is any vector, it is impossible that $x^n v \neq 0$ for all n, since x shifts V_μ to $V_{\mu+2}$, and $V_\mu = 0$ if $\mathrm{re}(\mu) > \mathrm{re}(\lambda)$. (The vector v may not be an H-eigenvector, but it can be decomposed as a sum of H-eigenvectors.) Now if $v \neq 0$ then there is a largest $n \geqslant 0$ such that $x^n v \neq 0$. Then $x^n v$ is a maximal vector.

(b) The "Verma module" $M(\lambda)$ has the following universal property. Note that for general semisimple Lie groups, the module $M(\lambda)$ is constructed by Humphreys in Section 20.3, and is called $Z(\lambda)$.

Proposition 3 (Universal Property of $M(\lambda)$)**.** *Let W be an $\mathfrak{sl}_2$-module containing a vector w_0 such that $xw_0 = 0$ and $hw_0 = \lambda w_0$. Then there is a unique L-module homomorphism $\phi : M(\lambda) \longrightarrow W$ such that $\phi(v_0) = w_\lambda$.*

Proof. Define $w_k = \frac{1}{k!} y^k w_\lambda$ $(k = 0, 1, 2, \cdots)$. Note that it is possible that $w_k = 0$ for k sufficiently large, so we do not assert that these are linearly independent. Because W is an $\mathfrak{sl}_2$-module,

$$hw_i = xyw_i - yxw_i, \qquad 2xw_i = hxw_i - xhw_i, \qquad -2yv_i = hyv_i - yhv_i.$$

Now we claim that the w_k satisfy the same relations

$$h(w_i) = (\lambda - 2i)w_i, \qquad y(w_i) = (i+1)w_{i+1} \tag{4}$$

$$xw_0 = 0, \qquad x(w_i) = (\lambda - i + 1)w_{i-1} \text{ if } i \geqslant 1, \tag{5}$$

as do the v_i (compare (2)). Since y shifts the eigenvalue of h by -2, the relation $h(w_i) = (\lambda - 2i)w_i$ is clear. Thus (4) is clear but (5) requires proof. We have assumed $xw_0 = xw_\lambda = 0$, and the other part needs proof.

We will prove this by induction, assuming

$$x(w_{i-1}) = (\lambda - i + 2)w_{i-2}. \tag{6}$$

(If $i = 1$, we interpret $w_{i-2} = w_{-1}$ to be zero.) Now since W is an $\mathfrak{sl}_2$-module, we have

$$hw_{i-1} = xyw_{i-1} - yxw_{i-1}$$

that is,

$$(\lambda - 2i + 2)w_{i-1} = xiw_i - y(\lambda - i + 2)w_{i-2}$$

where we have used (4) and in the second term, we have used (6). Continuing,

$$(\lambda - 2i + 2)w_i = xiw_i - (\lambda - i + 2)(i-1)w_{i-1}.$$

Rearranging,

$$ixw_i = ((\lambda - 2i + 2) + (\lambda - i + 2)(i-1))w_{i-1} = i(\lambda - i + 1)$$

and dividing by i proves (5).

Now since the v_i are a basis of $M(\lambda)$, there is a unique linear map $\phi : M(\lambda) \longrightarrow W$ mapping the basis vectors $v_i \in M(\lambda)$ to w_i. We argue that this is a Lie algebra homomorphism. Indeed, the fact that the w_i satisfy the relations (4) and (5) and the v_i satisfy the same relations (2) shows that this linear map commutes with the action of h, x, y. $\square$

Now we may prove (b). We assume that $\lambda + 1 = i$ is a positive integer. We want to define a homomorphism $M(\mu) \longrightarrow M(\lambda)$, and we will use $u_0, u_1, \cdots$ to denote the standard basis of $M(\mu)$ to avoid a conflict of notation with $v_0, v_1, \cdots$ which are the basis of $M(\lambda)$. Let $w_0 = v_i \in M(\lambda)$. Then $hw_0 = (\lambda - 2i)v_i = \mu w_0$ and the Proposition applies to give a homomorphism $M(\mu) \longrightarrow M(\lambda)$.

We have $yw_0 = yv_i = (\lambda - i + 1)v_{i-1} = 0$ since $\lambda - i + 1 = 0$, while $hw_0 = hv_i = (\lambda - 2i)v_i = \mu w_0$. Thus the Proposition applies (with μ instead of λ), and there is indeed a homomorphism $M(\mu) \longrightarrow M(\lambda)$ sending u_0 to v_i.

Section 8 #5. If L is semisimple, H a maximal toral subalgebra, prove that H is self-normalizing (i.e $H = N_L(H)$).

Solution. Let $X \in N(H)$, so $\mathrm{ad}(X)H \subset H$. We need to prove that $X \in H$. We recall that

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha, \qquad L_\alpha = \{y \in L | \mathrm{ad}(h)y = \alpha(h)y \text{ for } h \in H\}.$$

So we may write

$$X = X_0 + \sum_{\alpha \in \Phi} X_\alpha, \qquad X_0 \in H, X_\alpha \in L_\alpha.$$

We need to show that $X_\beta = 0$ for every root β. Find $h \in H$ such that $\beta(h) \neq 0$. (For example we could take $h = t_\beta$ or h_β.) Then $\mathrm{ad}(h)$ preserves the root space decomposition, and

$$\mathrm{ad}(h)X = \sum_{\alpha \in \Phi} \alpha(h)X_\alpha.$$

This is supposed to be in H, so every term must vanish. The coefficient of X_β is nonzero, so $X_\beta = 0$. This proves that if $X \in H$.

Section 8 #8. For $\mathfrak{sl}(n, F)$ calculate the root strings and Cartan integers. In particular prove that all Cartan integers $2(\alpha, \beta)/(\beta, \beta)$ with $\alpha \neq \beta$ for $\mathfrak{sl}(n)$ are $0, 1, -1$.

Solution. We will denote by $\mathfrak{h}$ the diagonal subalgebra of $\mathfrak{sl}(n)$, which Humphreys denotes H. The inner product is on $\mathfrak{h}^*$, and it is derived from the inner product on $\mathfrak{h}$ induced by the Killing form. Thus if $\lambda, \mu \in \mathfrak{h}^*$ then $(\lambda, \mu) = \kappa(t_\lambda, t_\mu)$ where $t_\lambda \in \mathfrak{h}$ is defined by the requirement that $\kappa(t_\lambda, H) = \lambda(H)$ for $H \in \mathfrak{h}$.

From Exercise 6 in Section 6 (HW3), any two associative bilinear forms on the simple Lie algebra $\mathfrak{sl}(n, F)$ are proportional. We may use the trace bilinear form, which induces a nondegenerate associative symmetric bilinear form on

$$\mathfrak{h} = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \right\}, \qquad \beta \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix} \right) = \sum t_i u_i. \tag{7}$$

This is proportional to the Killing form, and we do not need to compute the constant of proportionality (though this was computed in HW3). The reason we do not need to know the constant of proportionality is that the expression $2(\alpha, \beta)/(\beta, \beta)$ is unchanged if we multiply the inner product by a constant.

We will identify $\mathfrak{h}^*$ with F^n, in which $\lambda = (\lambda_1, \cdots, \lambda_n) \in F^n$ corresponds to the functional

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \longmapsto \sum_{i=1}^n \lambda_i t_i.$$

Then, since β corresponds to the usual dot product on $\mathfrak{h}$ by (7), we can also use the dot product on $\mathfrak{h}^* \cong F^n$, and

$$(\lambda, \mu) = \sum_{i=1}^n \lambda_i \mu_i. \tag{8}$$

Although this is not the inner product derived from κ, it is proportional, and we can use it to compute the Cartan constants. It will be convenient to denote by $\mathbf{e}_i$ ($i = 1, \cdots, n$) the standard basis of F^n, so $\mathbf{e}_i = (0, \cdots, 1, \cdots 0)$, where the 1 is in the i-th position. The roots Φ are vectors of the form $\mathbf{e}_i - \mathbf{e}_j$ where $i \neq j$. If $\alpha = \mathbf{e}_i - \mathbf{e}_j$ then for the inner product (8) we have $(\alpha, \alpha) = 2$, so the Cartan number $2(\alpha, \beta)/(\alpha, \alpha)$ is just (α, β). It is also clear that with $\beta \neq \pm\alpha$ the Cartan number $2(\alpha, \beta)/(\alpha, \alpha) = (\alpha, \beta)$ can be only $0, 1$ or -1.

Section 8 #11. If $(\alpha, \beta) > 0$, prove that $\alpha - \beta \in \Phi$ ($\alpha, \beta \in \Phi$). Is the converse true?

Solution. This is proved a little later in the book in Lemma 9.4. Here is another argument based on Proposition 8.4 (e). This says that if r, q are the largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots, then $\beta + i\alpha$ is a root for all integers i such that $-r \leqslant i \leqslant q$. In the case at hand, we know that β is a root and so, by Proposition 8.4 (c) is $\beta - \beta(h_\alpha)\alpha = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$. Note that $\beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ by Proposition 8.4 (c), and also $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} > 0$ since (α, β) is assumed positive. (The positivity of (α, α) is proved in Section 8.5.) Thus both 0 and $\beta(h_\alpha) > 0$ are integers in the interval $\{i \in \mathbb{Z} | \beta - i\alpha \in \Phi\}$, and since $\beta(h_\alpha)$ is a positive integer this means that $1 \in \{i \in \mathbb{Z} | \beta - i\alpha \in \Phi\}$. Therefore $\beta - \alpha$ is a root, and so is its negative $\alpha - \beta$.

The converse is *not* true. In the B_2 root system, here are the positive roots:

$$(1, -1), (0, 1), (1, 0), (1, 1).$$

If we take $\alpha = (1, 0)$ and $\beta(0, 1)$ then $\alpha - \beta$ is a root, but α and β are orthogonal. In the G_2 root system, we may find α and β such that $\alpha - \beta$ is a root but $(\alpha, \beta) < 0$.