# Math 210C Homework 3

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**Section 4 #7** Prove the converse to Theorem 4.3. That is, if $L \subseteq \mathfrak{gl}(V)$ is solvable prove that $\operatorname{tr}(xy) = 0$ for all $x \in [L, L]$ and $y \in L$.

**Solution.** By Lie's Theorem (Corollary A in Humphreys Section 4.1), we may find a basis of $V$ such that $L$ consists of upper triangular matrices. Then $[L, L]$ consists of upper triangular nilpotent matrices. With respect to this basis if $x \in [L, L]$ and $y \in L$ then $xy$ is upper triangular and nilpotent, so it has trace zero.

**Section 5 #1** Prove that if $L$ is nilpotent, the Killing form of $L$ is identically zero.

**Solution**. By Corollary 3.3 (Humphreys page 13) we may choose a basis of $L$ such that $\operatorname{ad}(L) \subseteq \operatorname{End}(L)$ consists of upper triangular nilpotent matrices. So if $x, y \in L$, then $\operatorname{ad}(x)$ and $\operatorname{ad}(y)$ are both upper triangular and nilpotent and so $\operatorname{ad}(x)\operatorname{ad}(y)$ is also upper triangular and nilpotent. Therefore $\kappa(x, y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)) = 0$.

**Section 5 #5** Let $L = \mathfrak{sl}(2, F)$. Compute the basis of $L$ dual to the standard basis, relative to the Killing form.

**Solution**. We will use this notation for the standard basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The matrices of $\operatorname{ad}(H)$, $\operatorname{ad}(E)$ and $\operatorname{ad}(F)$ with respect to the basis $E$, $H$, $F$ were already considered in HW1:

$$\operatorname{ad}(E) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{ad}(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \operatorname{ad}(F) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

from which we compute $\kappa$. For example

$$\kappa(E, F) = \operatorname{tr}\left(\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 4.$$

Here are the values:

$$\kappa(H, H) = 8, \qquad \kappa(E, F) = \kappa(F, E) = 4,$$

all other combinations such as $\kappa(H, E)$ are zero. Therefore the dual basis is given by the following table

| basis vector | $H$ | $E$ | $F$ |
|---|---|---|---|
| dual basis vector | $\frac{1}{8}H$ | $\frac{1}{4}F$ | $\frac{1}{4}E$ |

**Section 6 #6** Let $L$ be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on $L$. If $\beta, \gamma$ are nondegenerate, prove that $\beta$ and $\gamma$ are proportional.

[**Hint**: Use Schur's Lemma.]

**Solution**. If $V$ is an $L$-module, we make $V^*$ into a module by

$$(x \cdot \lambda)(v) = -\lambda(x \cdot v), \qquad \lambda \in V^*, v \in V. \tag{1}$$

To check this is a representation pf $L$, we need to show

$$[x, y] \cdot \lambda = x \cdot (y \cdot \lambda) - y \cdot (x \cdot \lambda). \tag{2}$$

Indeed

$$[x, y]\lambda(v) = -\lambda([x, y]v) = -\lambda(xyv - yxv) =$$
$$(x\lambda)(yv) - (y\lambda)(xv) = -(yx\lambda)(v) + (xy\lambda)(v)$$

proving (2). Thus $V^*$ is a module with the structure (1).

Now let $W$ be another module, and let $\beta : V \times W \longrightarrow F$ be a bilinear map that satisfies

$$\beta(xv, w) = -\beta(v, xw), \qquad x \in V, w \in W, x \in L.$$

Such a form is called *invariant*.

**Example 1.** *Take $V = W = L$, which is an $L$-module through the adjoint representation, with $\beta$ an associative bilinear form. Thus $x \cdot v$ means* $\mathrm{ad}(x)v = [x, v]$ *for these modules. Then the invariance property means*

$$\beta([x, v], w) = -\beta(v, [x, w]),$$

*which is equivalent to the form being associative in this example.*

**Lemma 2.** *If $V$ and $W$ are irreducible modules for a Lie algebra, and $\beta, \gamma : V \times W \longrightarrow F$ are invariant bilinear forms then $\beta, \gamma$ are prpoportional.*

*Proof.* Now let $\beta : V \times W \longrightarrow F$ be an invariant bilinear form. Define $\phi = \phi_\beta : V \longrightarrow W^*$ by

$$\phi_\beta(x)(w) = \beta(x, w), \qquad x \in V, w \in W.$$

We prove that $\phi_\beta$ is a homomorphism $V \longrightarrow W^*$. Indeed

$$\phi_\beta(z \cdot v)(w) = \beta(z \cdot v, w) = -\beta(v, z \cdot w) = -\phi_\beta(v)(z \cdot w) = (z \cdot \phi_\beta(v))(w).$$

This is true for all $w \in W$, proving that

$$\phi(z \cdot v) = z \cdot \phi(v)$$

as linear functionals on $W$ and therefore $\phi$ is a module homomorphism.

Now suppose that $V$ and $W$ are irreducible, so $W^*$ is irreducible. Then $\mathrm{Hom}_L(V, W^*)$ is one-dimensional by Schur's Lemma. So if $\beta, \gamma$ are invariant bilinear forms then $\phi_\beta$ and $\phi_\gamma$ are proportional. If $\phi_\gamma = c\phi_\beta$ then clearly $\gamma = c\beta$, so they are proportional. □

To solve the problem, let $V = W = L$ as in Example 1. Because we are assuming that $L$ is simple, it is irreducible as an $L$-module: indeed, a submodule would be an ideal, so the only submodules are 0 and $L$ itself. Therefore associative bilinear forms are proportional by the Lemma.

**Section 6 #7** It will be seen later that $\mathfrak{sl}(n, F)$ is actually *simple*. Assuming this and using Exercise 6, prove that the Killing form $\kappa$ on $\mathfrak{sl}(n, F)$ is related to the ordinary trace bilinear form by $\kappa(x, y) = 2n\text{tr}(xy)$.

**Solution**. Let $L = \mathfrak{sl}(n, F)$. By the previous exercise, the two associative bilinear forms in question are equivalent. Let $\tau : L \times L \longrightarrow F$ be the trace bilinear form $\tau(x, y) = \text{tr}(xy)$. To compute the constant of proportionality, let us take

$$H = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

and compute $\kappa(H, H)$. We will denote by $E_{i,j}$ the elementary matrix with an $i$ in the $i$-th position, zeros elsewhere. The *nonzero* eigenvalues of $\text{ad}(H)$ are $2, 1$ and $-1$, and the eigenspaces look like this. We will take $n = 5$ in laying out the eigenspaces for definiteness.

| Eigenvalue | Eigenspace ($n = 5$) | Eigenspace dimension |
|---|---|---|
| 2 | $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | 1 |
| $-2$ | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | 1 |
| 1 | $\begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \end{pmatrix}$ | $2(n-2)$ |
| $-1$ | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{pmatrix}$ | $2(n-2)$ |

From this data

$$\kappa(H, H) = \text{tr}(\text{ad}(H)^2)1 \times 2^2 + 1 \times (-2)^2 + 2(n-2) \times 1^2 + 2(n-2) \times (-1)^2 = 4n.$$

On the other hand $\tau(H, H) = \text{tr}(H^2) = 2$. So the constant of proportionality must be $2n$.