Math 210C Homework 2 Solutions

- Humphreys Section 2 (page 9) #1,7
- Section 3 (page 14) #1,2,6
- Section 4 (page 20) #1,5

Problem 1: Humphreys Section 2 #1. Prove that the set of all inner derivations $\mathrm{ad}(x)$, $x \in L$ is an ideal of $\mathrm{Der}(L)$.

Note: Humphreys uses this notation: $\mathfrak{t}(n, F)$ is the subalgebra of $\mathfrak{gl}(n, F)$ consisting of upper triangular matrices, $\mathfrak{n}(n, F)$ =strictly upper triangular matrices, and $\mathfrak{d}(n, F)$ =diagonal matrices. In the lectures I am using the following notations, which are more widely used.

Humphreys	Us	Common name
$\mathfrak{t}(n, F)$	$\mathfrak{b}$	"Borel subalgebra"
$\mathfrak{n}(n, F)$	$\mathfrak{n}$ or $\mathfrak{n}^+$	
$\mathfrak{d}(n, F)$	$\mathfrak{h}$	"Cartan subalgebra"

Problem 2: Section 2 #7. Prove that $\mathfrak{t}(n, F)$ and $\mathfrak{d}(n, F)$ are self-normalizing subalgebras of $\mathfrak{gl}(n, F)$, whereas $\mathfrak{n}(n, F)$ has normalizer $\mathfrak{t}(n, F)$.

Solution. Let $\mathfrak{g} = \mathfrak{gl}(n, F)$. We will denote by E_{ij} the elementary matrix with 1 in the i, j position and 0's elsewhere.

First let us find the normalizer of $\mathfrak{t}(n, F)$, which we denote $\mathfrak{b}$. The normalizer is by definition

$$N(\mathfrak{b}) = \{X \in \mathfrak{g} | \, \mathrm{ad}(X)\mathfrak{b} \subseteq \mathfrak{b}\}.$$

Clearly $\mathfrak{b} \subseteq N(\mathfrak{b})$. To prove the converse inclusion, let $X \in N(\mathfrak{b})$. We want to show that X is upper triangular. Suppose on the contrary that a matrix entry $X_{ij} \neq 0$ with $i > j$. Let $Y = E_{ii} \in \mathfrak{b}$. Then

$$[X, Y] = XY - YX$$

It is easy to see that YX is the i-th row of X, by which we mean, the matrix that has the same i-th row as X and 0's in every other row. So $[X, Y]$ is the i-th row of X minus the i-th column. In particular the i, j entry of $[X, Y]$ is $-X_{ij} \neq 0$. As $Y \in \mathfrak{b}$ this proves that $X \notin N(\mathfrak{b})$. We have proved that $N(\mathfrak{b}) = \mathfrak{b}$.

Now let $\mathfrak{h}$ denote $\mathfrak{d}(n, F)$ be the diagonal subalgebra. We claim that $N(\mathfrak{h}) = \mathfrak{h}$. Suppose that $X \in N(\mathfrak{h})$. If $X \notin \mathfrak{h}$ then $X_{ij} \neq 0$ for some $i \neq j$. Let $H = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \in \mathfrak{h}$. It is easy to see that $[X, H]_{ij} = (\lambda_j - \lambda_i)X_{ij}$, so if we choose $\lambda_i \neq \lambda_j$ then $[X, H] \notin \mathfrak{h}$. This means that $X \notin N(\mathfrak{h})$. We have then proved $N(\mathfrak{h}) = \mathfrak{h}$.

It is easy to check that $\mathfrak{h} \subseteq N(\mathfrak{n})$, so $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h} \subseteq N(\mathfrak{n})$. To prove that $N(\mathfrak{n}) \subseteq \mathfrak{b}$, suppose that $X \notin \mathfrak{b}$, so that $X_{ij} \neq 0$ for some $i > j$. Let $Z = E_{ji}$. Then $Z \in \mathfrak{n}$ but $[X, Z] \notin \mathfrak{n}$ since it has nonzero diagonal entries in the i and j positions, so $Z \notin N(\mathfrak{n})$.

Problem 3: Section 3 #1. Let I be an ideal of L. Then each member of the derived series or descending central series of I is an ideal of L.

Solution. Let us prove the following fact.

Lemma 1. *Let I, J be ideals of L. Then $[I, J]$ is an ideal of L.*

Proof. By definition, $[I, J]$ is the ideal generated by commutators $[x, y]$ with $x \in I$ and $y \in J$. To prove this is an ideal, with $z \in L$ we have

$$[z, [x, y]] = [[z, x], y] + [x, [z, y]] \in [I, J]$$

since $[z, x] \in I$ and $[z, y] \in J$. Thus $[L, [I, J]] \subseteq [I, J]$ proving that $[I, J]$ is an ideal. □

Now the derived series is

$$L^{(0)} = L, \qquad L^{(1)} = [L^{(0)}, L^{(0)}], \qquad L^{(2)} = [L^{(1)}, L^{(1)}], \tag{1}$$

and applying the Lemma repeatedly, each term is an ideal. The descending central series is handled by the same Lemma.

Problem 4: Section 3 #2. Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.

Solution. Humphreys defines L to be solvable if the derived series (1) terminates at 0, that is, $L^{(k)} = 0$ for sufficiently large k.

Lemma 2. *If L is a Lie algebra and M an ideal, then L/M is abelian if and only if $M \supset [L, L]$.*

Proof. Let $x, y \in L$ and let $\overline{x}$, $\overline{y}$ be the images in L/M. Clearly $[x, y] \in M$ if and only if $[\overline{x}, \overline{y}] = 0$ in L/M. From this, the conclusion is obvious. □

With the Lemma in mind, if L is solvable, we may take $L_i = L^{(i)}$ and we see that the condition stated in the problem is satisfied. Conversely, if the condition is satisfied, then since L/L_1 is abelian, we must have $L_1 \supset [L, L] = L^{(1)}$, and repeating this reasoning inductively we have $L_i \supset L^{(i)}$ for all i. We are assuming $L_k = 0$, so $L^{(k)} = 0$ and so L is solvable.

Problem 5: Section 3 #6. Prove that a sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore L possesses a unique maximal nilpotent ideal. (Humphreys asks you to determine this for particular algebras but you may skip this part.)

Solution. Let I and J be nilpotent ideals. We will denote by I^k the terms of the descending central series, so $I^0 = I$ and $[I, I^k] = I^{k+1}$. Let

$$F_k = I^k + (I^{k-1} \cap J) + (I^{k-2} \cap J^2) \cap \cdots \cap J^k.$$

We will argue that $(I + J)^k \subseteq F_k$. If $k = 1$, this true since clearly $I + J \subseteq F_1$. Since $[I, I^k] \subseteq I^{k+1}$, $[I, I^{k-i} \cap J^i] \subseteq I^{k+1-i} \cap J^i$ and $[I, J^i] \subseteq I \cap J^i$ we have $[I, F_k] \subseteq F_{k+1}$. Similarly $[J, F_k] \subseteq F_{k+1}$. Thus $[I + J, F^k] \subseteq F^{k+1}$. Our claim that $(I + J)^k$ then follows by induction.

Since $I^k = J^k = 0$ when $k \geqslant N$, we have $F_k = 0$ if $k \geqslant 2N$, proving that $(I + J)^k = 0$. Hence $I + J$ is a nilpotent ideal.

Now let I be a maximal nilpotent ideal. Then I contains every nilpotent ideal J, since $I + J$ is a nilpotent ideal, hence $I + J = I$ by the maximality of I.

Problem 6: Section 4 #1. Let $L = \mathfrak{sl}(V)$. Use Lie's Theorem to prove that $\mathrm{Rad}(L) = Z(L)$; conclude that L is semisimple. (See book for hint.)

Solution. Remember that $\mathrm{Rad}(L)$ is the maximal solvable ideal. By Lie's theorem $\mathrm{Rad}(L)$ stabilizes a flag $V = V_n \supset V_{n-1} \supset \cdots \supset V_0 = 0$ where $n = \dim(V)$ and $\dim(V_i) = i$. Choosing a basis $v_1, \cdots, v_n$ so that $v_i \in V_i - V_{i-1}$ with respect to this basis $\mathrm{Rad}(L)$ consists of upper triangular matrices. We will identify $L = \mathfrak{sl}(V) = \mathfrak{sl}(n, F)$ via this basis.

Now let us show that any element of $\mathrm{Rad}(L)$ is actually diagonal. There are different ways to proceed here, but we will use an automorphism $\theta : \mathfrak{sl}(n) \longrightarrow \mathfrak{sl}(n)$ defined by $\theta(X) = -{}^tX$. It is easy to check that this is an automorphism. Any automorphism of L must take $\mathrm{Rad}(L) \longrightarrow \mathrm{Rad}(L)$, so $\mathrm{Rad}(L)$ also consists of lower triangular matrices. In conclusion, every element of L is a diagonal matrix.

Now if $H \in \mathrm{Rad}(L)$ we will show that H is a scalar matrix. Indeed, let $H = \mathrm{diag}(a_1, \cdots, a_n)$. Then $[H, E_{ij}] \in \mathrm{Rad}(L)$ since $\mathrm{Rad}(L)$ is an ideal. But $[H, E_{ij}] = (a_i - a_j)E_{ij}$. This must be diagonal since we have already proved that every element of $\mathrm{Rad}(L)$ is diagonal, so $a_i = a_j$. We conclude that $H \in Z(L)$, the space of scalar matrices.

We have proved $\mathrm{Rad}(L) \subseteq Z(L)$. The other inclusion is clear since $Z(L)$ is a solvable ideal.

Problem 7: Section 4 #5. If $x, y \in \mathrm{End}(V)$ commute, prove that $(x + y)_s = x_s + y_s$ and $(x + y)_n = x_n + y_n$. Show by example that this can fail if x, y fail to commute. (See book for hint.)

Solution. Note that x commutes with y_s and y_n since it commutes with y and y_s, y_n are polynomials in y. Similarly y commutes with x_s and x_n, and indeed x, y, x_s, x_n, y_s, y_n all commute.

Lemma 3. *(i) If A and B are commuting semisimple matrices, then $A+B$ is also semisimple.*

(ii) If A and B are commuting nilpotent matrices, then $A + B$ is also nilpotent.

Proof. (i) We may extend the ground field to an algebraically closure, so A is diagonalizable. Every eigenspace of A is invariant under B, from which it follows that A and B can be simultaneously diagonalized. Then it is clear that $A + B$ is diagonalizable.

(ii) If $A^n = B^m = 0$ then by the binomial theorem, which applies since A and B commute, we have $(A + B)^{n+m} = 0$. □

Now $x_s + y_s$ is semisimple by the Lemma, and $x_n + y_n$ is nilpotent. They commute, and their sum is $x + y$, so by the uniqueness in Proposition 4.2 (a), we have

$$(x + y)_s = x_s + y_s, \qquad (x + y)_n = x_n + y_n.$$