Math 210C Homework 1 Solutions

- Humphreys Section 1 (pages 5-6) # 3,8,10 ($\mathtt{B}_2 \cong \mathtt{C}_2$ only).

Note: Problem 10 may be difficult at this stage in the book. As an alternative, you can substitute any other problem from Chapter 1.

Section 1 Problem 3. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}(2, F)$. Compute the matrices of $\mathrm{ad}(h)$, $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ relative to this basis.

Solution. We have

$$\mathrm{ad}(x)x = [x, x] = 0, \qquad \mathrm{ad}(x)h = [x, h] = -2x, \qquad \mathrm{ad}(x)y = h,$$

so with respect to this basis the matrix of $\mathrm{ad}(x)$ is

$$\mathrm{ad}(x) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and similarly

$$\mathrm{ad}(h) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}, \qquad \mathrm{ad}(y) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Section 1 Problem 8. Verify the stated dimension $2\ell^2 - \ell$ of $\mathtt{D}_\ell$.

Solution. By definition, this Lie algebra (also denoted $\mathfrak{so}(2\ell)$) consists of matrices X that satisfy $XJ = -J({}^tX)$ where

$$J = \begin{pmatrix} & I_\ell \\ I_\ell & \end{pmatrix}.$$

It is worth noting that if $X \in \mathtt{D}_\ell$ then so is ${}^tX$. This may be seen by conjugating the identity $XJ = -J({}^tX)$ by J and rearranging to obtain ${}^tXJ = -JX$.

We write X in block form as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The condition is that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} & I_\ell \\ I_\ell & \end{pmatrix} = - \begin{pmatrix} & I_\ell \\ I_\ell & \end{pmatrix} \begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix},$$

or

$$\begin{pmatrix} B & A \\ D & C \end{pmatrix} = - \begin{pmatrix} {}^tB & {}^tD \\ {}^tA & {}^tC \end{pmatrix}.$$

This gives us the identities $B = -{}^tB$, $C = -{}^tC$, and $D = -{}^tA$. There are l^2 entries in A, which determine the entries in D. Since B and C are skew-symmetric, they each contain $\frac{1}{2}\ell(\ell-1)$ independent entries. The dimension of the space of solutions is

$$\frac{1}{2}\ell(\ell-1) + \frac{1}{2}\ell(\ell-1) + \ell^2 = 2\ell^2 - \ell.$$

Section 1, Problem 10. For small values of ℓ, isomorphisms occur among certain of the classical algebras. Show that $\mathtt{B}_2 \cong \mathtt{C}_2$.

As I mentioned, this may be a hard problem placed so early in the book. I will give two solutions, using different ideas. One solution uses *roots* to figure out the correspondence. The other uses the exterior power of a representation.

Solution 1. We will try to solve this systematically, emphasizing ideas that will be important later. We will assume that an isomorphism $\phi : \mathtt{C}_2 \longrightarrow \mathtt{B}_2$ solution exists, and obtain formulas for it on a basis of $\mathtt{C}_2$. Once one has formulas for ϕ, one may check that it actually is an isomorphism, but we will omit this verification.

Humphreys defines $\mathtt{C}_\ell$ to be the Lie algebra of $X \in \mathfrak{gl}(4)$ such that

$$JX = -{}^tXJ, \qquad J = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

Note: I am writing ${}^tX$ instead of X^t for the transpose of a matrix. The matrix I am denoting J is denoted s in Humphreys.

Let us write X as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are $\ell \times \ell$ block matrices. Then the condition becomes

$$\begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix} \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = \begin{pmatrix} {}^tC & -{}^tA \\ {}^tD & -{}^tB \end{pmatrix},$$

so

$$X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}, \qquad B, C \text{ symmetric.}$$

Thus if $\ell = 2$, we obtain the following form for a typical element the Lie algebra $\mathtt{C}_2$:

$$\begin{pmatrix} a & b & t & u \\ c & d & u & v \\ x & y & -a & -c \\ y & z & -b & -d \end{pmatrix}.$$

On the other hand, Humphreys defines $\mathtt{B}_\ell$ to be the Lie subalgebra of $\mathfrak{gl}(5)$ such that

$$sX = -{}^tXs, \qquad s = \begin{pmatrix} 1 & & \\ & & I_\ell \\ & I_\ell & \end{pmatrix}.$$

This leads to the following form for X.

$$\begin{pmatrix} 1 & & \\ & & I_\ell \\ & I_\ell & \end{pmatrix} \begin{pmatrix} a & B_1 & B_2 \\ C_1 & M & N \\ C_2 & P & Q \end{pmatrix} = -\begin{pmatrix} a & B_1^t & B_2^t \\ B_1^t & M^t & P^t \\ B_2^t & N^t & Q^t \end{pmatrix} \begin{pmatrix} 1 & & \\ & & I_\ell \\ & I_\ell & \end{pmatrix}$$

$$\begin{pmatrix} a & B_1 & B_2 \\ C_2 & P & Q \\ C_1 & M & N \end{pmatrix} = -\begin{pmatrix} a & B_2^t & B_1^t \\ B_1^t & P^t & M^t \\ B_2^t & Q^t & N^t \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & B_1 & B_2 \\ -{}^tB_2 & M & N \\ -{}^tB_1 & P & -{}^tM \end{pmatrix}, \qquad N, P \text{ skew-symmetric.}$$

Note that a is a 1×1 matrix, B_1 and B_2 are $1 \times \ell$ matrices and M, N, P are $\ell \times \ell$.

For $\ell = 2$, by a similar computation, we obtain the following form for a typical form for the Lie algebra of $\mathtt{B}_2$:

$$\begin{pmatrix} 0 & \alpha & \beta & \gamma & \delta \\ -\gamma & \varepsilon & \eta & 0 & \lambda \\ -\delta & \zeta & \theta & -\lambda & 0 \\ -\alpha & 0 & \mu & -\varepsilon & -\zeta \\ -\beta & -\mu & 0 & -\eta & -\theta \end{pmatrix}.$$

To construct an isomorphism, we must make some choices. For although there is *essentially* only one isomorphism $\phi : \mathtt{C}_2 \longrightarrow \mathtt{B}_2$, "essentially" means unique up to conjugation. So given one isomorphism, we may conjugate it by any element of the symplectic group $\mathrm{Sp}(4)$ to obtain another. Thus to pin down one isomorphism, we must make some choices.

The first choice is that the diagonal subalgebras (called *Cartan subalgebras* later in the book) correspond. These are the abelian Lie algebras:

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & & & \\ & d & & \\ & & -a & \\ & & & -d \end{pmatrix} \right\}, \qquad \mathfrak{t} = \left\{ \begin{pmatrix} 0 & & & & \\ & \varepsilon & & & \\ & & \theta & & \\ & & & -\varepsilon & \\ & & & & -\theta \end{pmatrix} \right\} \tag{1}$$

So we hope that

$$\phi \begin{pmatrix} a & & & \\ & d & & \\ & & -a & \\ & & & -d \end{pmatrix} = \begin{pmatrix} 0 & & & & \\ & \varepsilon & & & \\ & & \theta & & \\ & & & -\varepsilon & \\ & & & & -\theta \end{pmatrix}$$

for some ε, θ, but we need to figure out how ε, θ depend on a and d. To figure this out, let us decompose $\mathtt{C}_2$ into one-dimensional eigenspaces under $\mathrm{ad}(\mathfrak{h})$.

Definition 1. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ an abelian subalgebra. A root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a nonzero linear functional α on $\mathfrak{h}$ such that there exists a vector X_α such that*

$$\mathrm{ad}(H)X_\alpha = \alpha(H)X_\alpha \qquad \text{for all } H \in \mathfrak{h}.$$

The set of roots is called the root system.

So with $\mathfrak{h} \subset \mathtt{C}_2$ and $\mathfrak{t} \subset \mathtt{B}_2$ as in (1) let us compute the root systems.

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We compute

$$\mathrm{ad}(H)X_1 = [H, X_1] = (a-d)X_1, \qquad H = \begin{pmatrix} a & & & \\ & d & & \\ & & -a & \\ & & & -d \end{pmatrix}, \tag{2}$$

so this an eigenvector for the linear functional $H \mapsto a - d$. Such linear functionals of the Cartan subalgebra (called *roots*) are useful for solving this particular problem. We find the following $\mathrm{ad}(\mathfrak{h})$ eigenvectors:

X_α	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\alpha(H)$	$a - d$	$2d$	$a + d$	$2a$
X_α	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\alpha(H)$	$-(a - d)$	$-2d$	$-a - d$	$-2a$

Hence the root system of $\mathtt{C}_2$ is the set of roots

$$\Phi\left(\mathtt{C}_2\right) = \{a-d, 2d, a+d, 2a, -(a-d), -2d, 2a, -(a+d)\}$$

which are all linear functionals on the matrix $H \in \mathfrak{h}$ in (2). Note that $\mathtt{C}_2$ is the direct sum of $\mathfrak{h}$ and the eight one-dimensional vectors X_α ($\alpha \in \Phi\left(\mathtt{C}_2\right)$).

Now we perform the same calculation for $\mathtt{B}_2$ with respect to the Cartan subalgebra $\mathfrak{t}$. Denote

$$T = \begin{pmatrix} 0 & & & & \\ & \varepsilon & & & \\ & & \theta & & \\ & & & -\varepsilon & \\ & & & & -\theta \end{pmatrix} \in \mathfrak{t}.$$

We are now looking for nonzero linear functions $\beta \in \mathfrak{t}^*$ (roots) and vectors $Y_\beta \in \mathtt{B}_2$ such that

$$[T, Y_\beta] = \beta(T)Y_\beta, \qquad T \in \mathfrak{t}.$$

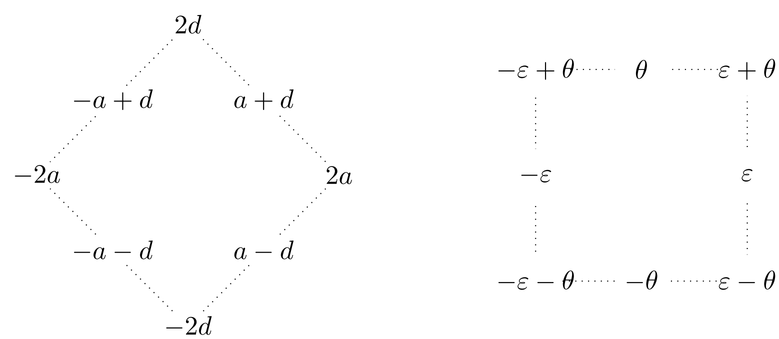
We find the following roots:

Y_β	$\begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ -1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&-1&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$
$\beta(T)$	θ	$\varepsilon-\theta$	ε	$\varepsilon+\theta$
Y_β	$\begin{pmatrix} 0&0&-1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&-1&0&0&0 \end{pmatrix}$
$\beta(T)$	$-\theta$	$-(\varepsilon-\theta)$	$-\varepsilon$	$-(\varepsilon+\theta)$

Thus the root system is

$$\Phi(\mathtt{B}_2) = \{\theta, \varepsilon-\theta, \varepsilon, \varepsilon+\theta, -\theta, -(\varepsilon-\theta), -\varepsilon, -(\varepsilon+\theta)\}.$$

Now we can start to construct the isomorphism $\phi : \mathtt{C}_2 \longrightarrow \mathtt{B}_2$. We have assumed that ϕ will take the Cartan subalgebra $\mathfrak{h}$ of $\mathtt{C}_2$ to the Cartan subalgebra $\mathfrak{t}$ of $\mathtt{B}_2$. The root systems must correspond under this correspondence. It will be helpful to visualize them.



We can map $\mathfrak{h} \longrightarrow \mathfrak{t}$ in such a way that the roots $a-d$ and $2d$ correspond to θ and $\varepsilon-\theta$. Solving for ε we have

$$\theta = a-d, \qquad \varepsilon = a+d,$$

so on $\mathfrak{h}$, we see that ϕ must be the map

$$\phi\begin{pmatrix} a&&&\\ &d&&\\ &&-a&\\ &&&-d \end{pmatrix} = \begin{pmatrix} 0&&&&\\ &a+d&&&\\ &&a-d&&\\ &&&-a-d&\\ &&&&-a+d \end{pmatrix}. \tag{3}$$

Once we see that we can map the Cartan subalgebras isomorphically so that the roots correspond, it follows that $\mathtt{B}_2 \cong \mathtt{C}_2$ from Theorem 14.2 of Humphreys (page 75). However since this is later in the book, we will give some more details.

Note that the $\mathtt{C}_2$ Lie algebra is spanned by $\mathfrak{h}$ and the eight root vectors X_α ($\alpha \in \Phi(\mathtt{C}_2)$). We have already determined ϕ on $\mathfrak{h}$ by (3). So we need to compute $\phi(X_\alpha)$. We can almost do this immediately. Indeed, if α is a root of $\mathtt{C}_2$ and β is the corresponding root of $\mathtt{B}_2$ we need $\phi(X_\alpha) = c_\alpha Y_\beta$ for some constant c_α.

The constants c_α need to be chosen carefully, but we do have some freedom to adjust them, since we may conjugate ϕ by a matrix of the form

$$\begin{pmatrix} u & & & \\ & v & & \\ & & u^{-1} & \\ & & & v^{-1} \end{pmatrix} \in \mathrm{Sp}(4),$$

since this conjugation is easily seen to be an automorphism of $\mathtt{C}_2$. Using this flexibility, we can arrange that $c_\alpha = 1$ for two roots. We choose to make $c_{a-d} = c_{2d} = 1$. Thus

$$\phi(X_{a-d}) = Y_\theta, \qquad \phi(X_{2d}) = Y_{\varepsilon-\theta}. \tag{4}$$

Assuming this, we will deduce the following values for ϕ on the X_α:

X	X_{a-d}	X_{2d}	X_{a+d}	X_{2a}	$X_{-(a-d)}$	X_{-2d}	$X_{-(a+d)}$	X_{-2a}
$\phi(X)$	Y_θ	$Y_{\varepsilon-\theta}$	Y_ε	$\frac{1}{2}Y_{\varepsilon+\theta}$	$2Y_{-\theta}$	$Y_{-(\varepsilon-\theta)}$	$2Y_{-\varepsilon}$	$2Y_{-(\varepsilon+\theta)}$

The constants c_α that appear here (for example $c_{2a} = \frac{1}{2}$) were arrived at using commutation relations and checked with a computer program. For example, find that

$$[X_{a-d}, X_{2d}] = X_{a+d}$$

and since we have adjusted ϕ so that $\phi(X_{a-d}) = Y_\theta$, $\phi(X_{2d}) = Y_{\varepsilon-\theta}$ we must have

$$\phi(X_{a+d}) = [Y_\theta, Y_{\varepsilon-\theta}] = Y_\varepsilon.$$

Second Solution. Let V be a symplectic vector space over a field F of characteristic $\neq 2$. This means that V is a vector space equipped with a nondegenerate bilinear form $\beta : V \times V \longrightarrow F$ such that $\beta(x, y) = -\beta(y, x)$. This solution will be a little sketchy. We will construct a homomorphism from $\mathfrak{sp}(2n, F)$ to an odd orthogonal Lie algebra $\mathfrak{o}_\alpha(N, F)$ with respect a symmetric bilinear form α on an $N = n(2n-1)$ dimensional vector space. Then we will show that actually the image of this homomorphism factors through a slightly smaller orthogonal algebra $\mathfrak{o}_\alpha(N-1, F)$. If $n = 2$ then $N - 1 = 5$, so this homomorphism maps $\mathtt{C}_2 = \mathfrak{sp}(4)$ to $\mathtt{B}_2 = \mathfrak{so}(5)$, and in this case the homomorphism is an isomorphism.

We will not check that the orthogonal algebra $\mathfrak{o}_\alpha(N-1, F)$ is the version stabilizes the "split" symmetric bilinear form with matrix

$$\begin{pmatrix} 1 & & \\ & & I_2 \\ & I_2 & \end{pmatrix},$$

so in this respect this solution will be incomplete. Over an algebraically closed field, any symmetric bilinear form is equivalent to a split form, so the proof is complete over $\mathbb{C}$. But the first solution shows that this result is true over a general field, and certainly with a bit more work that could be proved in this second solution.

Let $W = V \wedge V$ be the exterior square. This has the following *universal property*: if $\phi : V \times V \longrightarrow U$ is any skew-symmetric bilinear map to a vector space W, then there exists a unique linear map $\Phi : V \wedge V \longrightarrow U$ such that $\phi(x, y) = \Phi(x \wedge y)$. (See Lang's *Algebra* page 732.)

Lemma 2. *There is a nondegenerate symmetric bilinear map*

$$\alpha : W \times W \longrightarrow F$$

such that

$$\alpha(w \wedge x, y \wedge z) = \beta(w, y)\beta(x, z) - \beta(w, z)\beta(x, y). \tag{5}$$

Proof. Let $y, z \in V$. Define $\alpha_{y,z} : V \times V \longrightarrow F$ by

$$\alpha_{y,z}(w, x) = \beta(x, y)\beta(w, z) - \beta(x, z)\beta(w, y).$$

This map is bilinear and skew-symmetric, so by the universal property of the exterior square it factors through $W = V \wedge V$. That is, there exists a unique linear map $\gamma_{y,z} : W \longrightarrow F$ such that $\alpha_{y,z}(w, x) = \gamma_{y,z}(w \wedge x)$. Then the map $V \times V \longrightarrow W^*$ defined by $y, z \mapsto \gamma_{y,z}$ is bilinear and skew-symmetric, so another application of the universal property of the exterior square shows there is a linear map $\lambda : W \longrightarrow W^*$ such that $\gamma_{y,z} = \lambda(y \wedge z)$. Define $\alpha : W \times W \longrightarrow F$ by

$$\alpha(\xi, \eta) = \lambda(\eta)\xi.$$

Then this map is bilinear and satisfies (5). The form α is symmetric since, using the fact that β is skew-symmetric, the right-hand side of (5) is unchanged on interchanging $w \wedge x$ with $y \wedge z$.

We need to show that α is nondegenerate. We will show that if $\tau : W \longrightarrow F$ is any linear functional, then there exists $\eta \in W$ such that $\tau(\xi) = \alpha(\xi, \eta)$. Consider the bilinear form $\theta : V \times V \longrightarrow F$ defined by $\theta(w, x) = \tau(w \wedge x)$. There exist ϕ_i, ψ_i $(i = 1, \cdots n)$ such that

$$\theta(w, x) = \sum_{i=1}^{n} \phi_i(w)\psi_i(x),$$

since any bilinear form on V has this form. Then since β is nondegenerate, we may find elements $y_i, z_i \in V$ such that $\phi_i(w) = \beta(w, y_i)$ and $\psi_i(x) = \beta(x, z_i)$. Then

$$\tau(w \wedge x) = \sum \beta(w, y_i)\beta_i(x, z_i).$$

On the other hand

$$\tau(w \wedge x) = -\tau(x \wedge w) = -\sum \beta(x, y_i)\beta_i(w, z_i).$$

Adding these two equations

$$2\tau(w \wedge x) = \beta(w, y_i)\beta_i(x, z_i) - \beta(x, y_i)\beta_i(w, z_i) = \sum_i \alpha(w \wedge x, y_i \wedge z_i).$$

Thus with $\eta = \frac{1}{2} \sum y_i \wedge z_i$ we see that $\tau(\xi) = \alpha(\xi, \eta)$. Because τ was an arbitrary element of W^*, this shows that α is nondegenerate. □

Let

$$\mathfrak{sp}_\beta(V) = \{X \in \mathrm{End}(V) | \beta(Xx, y) = -\beta(x, Xy)\}$$

and

$$\mathfrak{o}_\alpha(W) = \{X \in \mathrm{End}(W) | \alpha(X\xi, \eta) = -\alpha(x, X\eta)\}$$

be the symplectic and orthogonal Lie algebras associated with the forms β and α. We define an action of $\mathrm{Sp}_\beta(V)$ on W by $X(t \wedge u) = Xt \wedge u + t \wedge Xu$.

Lemma 3. *The form α is invariant under $X \in (V)$. Thus the endomorphism of W induced by X is in $\mathfrak{o}_\alpha(W)$.*

Proof. Indeed

$$\alpha(X(w \wedge x), y \wedge z) = \alpha(Xw \wedge x, y \wedge z) + \alpha(w \wedge Xx, y \wedge z) =$$

$$\beta(Xw, y)\beta(x, z) - \beta(Xw, z)\beta(x, y) + \beta(w, y)\beta(Xx, z) - \beta(w, z)\beta(Xx, y) =$$

$$-\beta(w, Xy)\beta(x, z) + \beta(w, Xz)\beta(x, y) - \beta(w, y)\beta(x, Xz) + \beta(w, z)\beta(x, Xy) =$$

$$-\alpha(w \wedge x, Xy \wedge z + y \wedge Xz) = -\alpha(w \wedge x, X(y \wedge z)).$$

□

Now there exists a vector $\xi_0 \in W$ such that $\beta(x, y) = \alpha(x \wedge y, \xi_0)$. Indeed, since β is skew-symmetric, there exists a linear functional on $W = V \times V$ that maps $x \wedge y \mapsto \beta(x, y)$. Then since α is nondegenerate this linear functional can be realized as the inner product with a vector ξ_0.

Lemma 4. *We have $X\xi_0 = 0$ for all $X \in \mathfrak{sp}(4)$.*

Proof. It is enough to show $\langle x \wedge y, X\xi_0 \rangle = 0$ for $x, y \in V$. We have

$$\langle x \wedge y, X\xi_0 \rangle = -\langle X(x \wedge y), \xi_0 \rangle = -\langle Xx \wedge y, \xi_0 \rangle - \langle x \wedge Xy, \xi_0 \rangle =$$

$$-\beta(Xx, y) - \beta(x, Xy) = 0.$$

□

Now let W_0 be the orthogonal complement of ξ_0 in W, a subspace of codimension 1. Then W_0 is invariant under the action of $\mathfrak{sp}(4)$, and the symmetric bilinear form α restricted to W_0 remains nondegenerate. It remains to be checked that the form α restricted to W_0 is "split," meaning equivalent to the form

$$\begin{pmatrix} 1 & & \\ & & I_2 \\ & I_2 & \end{pmatrix}$$

used to define $\mathtt{B}_2$. We omit this, but at least if the ground field F is algebraically closed, any two nondegenerate symmetric bilinear forms are equivalent.