Math 210C Homework 7

- Section 12 (p.67) # 4
- Section 13 (p.72) # 7 (explain why relevant)
- Section 22 (p.126) # 7
- Section 24 (p.141) # 1

Section 12 #4. Prove that the long roots in G_2 form a root system in E of type A_2.

Remark: One could also say that the *short* roots of G_2 form a root system of type A_2. But there is a difference: If α, β are long roots in $\Phi = \Phi_{G_2}$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta$ is a long root. This implies that the x_α (α long) generate a Lie subalgebra of the Lie algebra $\mathtt{G}_2$ isomorphic to A_2, because ($\mathfrak{h}$ denoting the Cartan subalgebra)

$$\mathfrak{h} \oplus \bigoplus_{\alpha \text{ long}} F x_\alpha$$

is closed under [,]. This would fail for short roots.

Section 13 #7. If $\varepsilon_1, \cdots, \varepsilon_\ell$ is an obtuse basis of the Euclidean space E (i.e., all $(\varepsilon_i, \varepsilon_j) \leqslant 0$ for $i \neq j$), prove that the dual basis is *acute* (i.e. $(\varepsilon_i^*, \varepsilon_j^*) \geqslant 0$ for $i \neq j$). Explain briefly how this is relevant to root systems.

[**Hint:** reduce to the case $\ell = 2$.]

Section 22 #7. Let $L = \mathfrak{sl}(2, F)$, and identify $m\lambda_1$ with the integer m. Use Propositions A and B of (22.5), along with Theorem 7.2 to derive the *Clebsch-Gordan formula*: if $n \leqslant m$, tnen

$$V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n).$$

(Compare Exercise 7.6).

Section 24 #1. Give a direct proof of Weyl's character formula (24.3) for type $\mathtt{A}_1$.