Homework 5 Solutions

May 27, 2022

Problem 1: Section 9 (page 45) #8. Compute the root strings in G_2 to verify the relation $r - q = \langle \beta, \alpha \rangle$.

Note: The notation $\langle \beta, \alpha \rangle$ is introduced on page 42 of Humphreys. It means

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \beta(h_\alpha) = (\alpha^\vee, \beta)$$

where $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Note that if we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the inner product (,), then $\alpha^\vee \in \mathfrak{h}^*$ is really the same as $h_\alpha \in \mathfrak{h}$. Either one is called a *coroot*.

Solution. Let α_1 be the short simple root and α_2 the long simple root of G_2. Thus the roots are $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}$. We will only compute the root strings for α_1 and α_2. This is sufficient to verify the relation $r - q = \langle \beta, \alpha \rangle$ because the Weyl group action permutes the roots and root strings, and can map any root α in G_2 to either α_1 or α_2. We may take

$$\alpha_1 = (0, 1, -1), \qquad \alpha_2 = (1, -2, 1)$$

embedded in $\{(t_1, t_2, t_3) \,|\, \sum t_i = 0\}$. The restriction of the usual inner product on $\mathbb{R}^3$ to this hyperplane

First let us consider the α_1 root strings.

β	α_1-root string	r	q	$\langle\beta,\alpha_1\rangle$	$\langle\beta,\alpha_1\rangle = r-q$
α_1	$\{-\alpha_1, 0, \alpha_1\}$	2	0	2	✓
α_2	$\{\alpha_2, \alpha_2+\alpha_1, \alpha_2+2\alpha_1, \alpha_2+3\alpha_1\}$	0	3	-3	✓
$\alpha_2+\alpha_1$	$\{\alpha_2, \alpha_2+\alpha_1, \alpha_2+2\alpha_1, \alpha_2+3\alpha_1\}$	1	2	-1	✓
$\alpha_2+2\alpha_1$	$\{\alpha_2, \alpha_2+\alpha_1, \alpha_2+2\alpha_1, \alpha_2+3\alpha_1\}$	2	1	1	✓
$\alpha_2+3\alpha_1$	$\{\alpha_2, \alpha_2+\alpha_1, \alpha_2+2\alpha_1, \alpha_2+3\alpha_1\}$	3	0	3	✓
$2\alpha_2+3\alpha_1$	$\{2\alpha_2+3\alpha_3\}$	0	0	0	✓
$-\alpha_1$	$\{-\alpha_1, 0, \alpha_1\}$	0	2	-2	✓
$-\alpha_2$	$\{-\alpha_2, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2, -3\alpha_1-\alpha_2\}$	3	0	3	✓
$-(\alpha_2+\alpha_1)$	$\{-\alpha_2, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2, -3\alpha_1-\alpha_2\}$	2	1	1	✓
$-(\alpha_2+2\alpha_1)$	$\{-\alpha_2, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2, -3\alpha_1-\alpha_2\}$	1	2	-1	✓
$-(\alpha_2+3\alpha_1)$	$\{-\alpha_2, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2, -3\alpha_1-\alpha_2\}$	0	3	-3	✓
$-(2\alpha_2+3\alpha_1)$	$\{-2\alpha_2-3\alpha_3\}$	0	0	0	✓

Next here are the α_2 root strings:

β	α_2-root string	r	q	$\langle\beta,\alpha_1\rangle$	$\langle\beta,\alpha_1\rangle = r-q$
α_1	$\{\alpha_1, \alpha_1+\alpha_2\}$	0	1	-1	✓
α_2	$\{\alpha_2, 0, -\alpha_2\}$	2	0	2	✓
$\alpha_2+\alpha_1$	$\{\alpha_1, \alpha_2+\alpha_1\}$	1	0	1	✓
$\alpha_2+2\alpha_1$	$\{\alpha_2+2\alpha_1\}$	0	0	0	✓
$\alpha_2+3\alpha_1$	$\{\alpha_2+3\alpha_1, 2\alpha_2+3\alpha_1\}$	0	1	-1	✓
$2\alpha_2+3\alpha_1$	$\{\alpha_2+3\alpha_1, 2\alpha_2+3\alpha_1\}$	1	0	-1	✓
$-\alpha_1$	$\{-\alpha_1, -\alpha_1-\alpha_2\}$	1	0	1	✓
$-\alpha_2$	$\{\alpha_2, 0, -\alpha_2\}$	0	2	-2	✓
$-(\alpha_2+\alpha_1)$	$\{-\alpha_1, -\alpha_1-\alpha_2\}$	0	1	-1	✓
$-(\alpha_2+2\alpha_1)$	$\{-\alpha_2-2\alpha_1\}$	0	0	0	✓
$-(\alpha_2+3\alpha_1)$	$\{-\alpha_2-3\alpha_1, -2\alpha_2-3\alpha_1\}$	1	0	1	✓
$-(2\alpha_2+3\alpha_1)$	$\{-\alpha_2-3\alpha_1, -2\alpha_2-3\alpha_1\}$	0	1	-1	✓

Problem 2: Section 10 (page 54) #4. Verify the Corollary of Lemma 10.2A directly for G_2.

The Corollary in question states:

Corollary 1. *Every $\beta \in \Phi^+$ can be written in the form $\alpha_1 + \ldots + \alpha_k$, $\alpha_i \in \Delta$, not necessarily distinct, in such a way that each partial sum $\alpha_1 + \ldots + \alpha_i$ is a root.*

We will modify the notation of the corollary by using α_i to denote the simple roots. (For Humphreys, Δ is the set of simple roots.)

Let us start by writing the highest root $3\alpha_1 + 2\alpha_2$ as

$$(\alpha_1) + (\alpha_2) + (\alpha_1) + (\alpha_1) + (\alpha_1) + (\alpha_2).$$

Now we note that the sum of every initial segment of this sum is a positive root. This verifies the assertion for $\beta = 3\alpha_1 + 2\alpha_2$. But in fact it verifies it for all positive roots, since we can write any positive root β as one of these partial sums, so an initial segment of the sequence $\{\alpha_1, \alpha_2, \alpha_1, \alpha_1, \alpha_2, \alpha_1\}$ will solve the problem for β.

Problem 3: Section 10 (page 54) #6. Define a function $\mathrm{sn} : W \longrightarrow \{\pm 1\}$ by $\mathrm{sn}(\sigma) = (-1)^{\ell(\sigma)}$. Prove that sn is a homomorphism.

Solution. By Lemma A in Section 10.3 $\ell(\sigma) = n(\sigma)$ where $n(\sigma)$ is number of $\beta \in \Phi^+$ such that $\sigma(\beta) \in \Phi^-$. Thus

$$\ell(\sigma) = |\Phi^- \cap \sigma(\Phi^+)|. \tag{1}$$

Now

Lemma 1. *We have*

$$\ell(\sigma\sigma_\beta) = \begin{cases} \ell(\sigma) + 1 & \text{if } \sigma(\beta) \in \Phi^+, \\ \ell(\sigma) - 1 & \text{if } \sigma(\beta) \in \Phi^-. \end{cases}$$

Proof. Note that $\sigma_\beta(\Phi)^+ = \Phi^+ \backslash \{\beta\} \cup \{-\beta\}$ by Lemma B in Section 10.2, since the simple reflection σ_β interchanges $\beta \in \Phi^+$ and $-\beta \in \Phi^-$, but otherwise permutes the positive roots. Using (1), $\ell(\sigma\sigma_\beta) = |\Phi^- \cap \sigma(\Phi^+ \backslash \{\beta\} \cup \{-\beta\})|$, or

$$\ell(\sigma\sigma_\beta) = |\Phi^- \cap (\sigma\Phi^+ \backslash \{\sigma(\beta)\} \cup \{-\sigma(\beta)\})|$$

If $\sigma(\beta) \in \Phi^+$ this means

$$\ell(\sigma\sigma_\beta) = |(\Phi^- \cap \sigma\Phi^+) \cup \{-\sigma(\beta)\}| = \ell(\sigma) + 1.$$

On the other hand if $\sigma(\beta) \in \Phi^-$,

$$\ell(\sigma\sigma_\beta) = |(\Phi^- \cap \sigma\Phi^+) \backslash \{\sigma(\beta)\}| = \ell(\sigma) - 1.$$

$\square$

In both cases $\mathrm{sgn}(\sigma\sigma_\beta) = -\mathrm{sgn}(\sigma)$. Therefore if σ is written as a product of k simple reflections, $\mathrm{sgn}(\sigma) = (-1)^k$. From this it is clear that $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$.

Problem 4: Section 12 (page 68) #5. In constructing C_ℓ would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.

Solution. This is correct if $\ell \leqslant 3$. However if $\ell \geqslant 4$, it fails because we have vectors such as $(1, 1, 1, 1)$ in $I = \mathbb{Z}^4$ which has length squared equal to 4, but is not a root.

Problem 5: Section 13 (page 71) #2. Show by example (e.g., for A_2) that $\lambda \notin \Lambda^+$, $\alpha \in \Delta$, $\lambda - \alpha \in \Lambda^+$ is possible.

Solution. We can always take $\lambda = \alpha$ so that $\lambda - \alpha = 0$ is a dominant weight, because only for type A_1 are the simple roots themselves dominant.

Problem 6: Section 13 (page 71) #9. Let $\lambda \in \Lambda^+$. Show that $\sigma(\lambda + \rho) - \rho$ is dominant only for $\sigma = 1$. (I am writing ρ for half the sum of the positive roots instead of δ.)

Solution: We will deduce this from Lemma A in Section 13.2, on page 68. Assume that λ is dominant and $\sigma(\lambda + \rho) - \rho$ is dominant. Then $\lambda + \rho$ and $\sigma(\lambda + \rho)$ are both strongly dominant. Therefore the two parts to the Lemma show that $\lambda + \rho = \sigma(\lambda + \rho)$ (since both are dominant elements of the same Weyl group orbit, and that $\sigma = 1$ (since $\lambda + \rho$ is strongly dominant).