Homework 4 Solutions

Problem 1. Section 7.2 (page 34) #2. $M = \mathfrak{sl}_3$ contains a copy of $L = \mathfrak{sl}_2$ in its upper right-hand corner. Write M as a direct sum of irreducible L-modules (M viewed as an L-module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Solution: Recall that $V(k)$ is the $k+1$ dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. With respect to the subalgebra

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

of $\mathfrak{sl}_3$, the space $\mathfrak{sl}_2$ itself is a submodule equivalent to the adjoint representation of $\mathfrak{sl}_2$, which is $V(2)$. There are two two dimensional invariant subspaces:

$$\left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -y & x & 0 \end{pmatrix} \right\}$$

These are both isomorphic to the standard module $V(1)$. We remark that if we did this for $\mathfrak{sl}_n \subset \mathfrak{sl}_{n+1}$ in general, there would still be two n-dimensional submodules realizing the standard module and its dual (contragredient) module. For $\mathfrak{sl}_2$, the standard module and its dual are isomorphic, but not for $\mathfrak{sl}_n$ with $n > 2$. Finally, there is a one dimensional module

$$\mathbb{C} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

which gives a $V(0)$. Thus $\mathfrak{sl}_3 \cong V(0) \oplus 2V(1) \oplus V(2)$ as an $\mathfrak{sl}_2$-module.

Problem 2. Section 7.2 (page 34) #6. Decompose the tensor product of two L-modules $V(3)$ and $V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

Solution: We could ask for an explicit decomposition of $V(3) \otimes V(7)$ into irreducible subspaces, but instead I will just prove that the decomposition is correct, and also generalize it. Let q be an indeterminate. Define the character of a representation V to be

$$\chi_V = \sum_k \dim(V_k) q^k$$

where we recall the notation used in Theorem 7.2 that V_k is the k-eigenspace of $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ (denoted h by Humphreys). So by Theorem 7.2

$$\chi_{V(k)} = q^k + q^{k-2} + \ldots + q^{-k} = \frac{q^{k+1} - q^{-(k+1)}}{q - q^{-1}}.$$

The characters of the irreducibles are linearly independent because the numerators $q^{k+1} - q^{-(k+1)}$ $(k \in \mathbb{Z})$ are linearly independent. Since we know that every finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ decomposes uniquely into a direct sum of irreducible,s the linearly independence of the characters of irreducibles clearly implies that the representation is determined by its characters

Now the character of $V(3) \otimes V(7)$ is the product of the characters, and it is good to expand the character of the smaller $V(3)$, and leave the character of the $V(7)$ in the unexpanded form. Thus

$$\chi_{V(3)\otimes V(7)} = \chi_{V(3)} \cdot \chi_{V(7)} = (q^3 + q + q^{-1} + q^{-3})(q - q^{-1})^{-1}(q^8 - q^{-8})$$

$$= (q - q^{-1})^{-1}(q^{11} - q^{-11} + q^9 - q^{-9} + q^7 - q^{-7} + q^5 - q^{-5}) = \chi_{V(10)} \oplus \chi_{V(8)} \oplus \chi_{V(6)} + \chi_{V(4)}.$$

This prove that $V(3) \circ V(7) \cong V(10) \oplus V(8) \oplus V(6) \oplus V(4)$.

We can generalize this as follows. Without loss of generality, assume that $m < n$. Then the character of $V(m) \otimes V(n)$ can be written

$$\left(\sum_{j=0}^{m} q^{m-2j} \right) (q - q^{-1})^{-1}(q^{n+1} - q^{-n-1}) =$$

$$(q - q^{-1})^{-1} \left(q^{n+1} \sum_{j=0}^{m} q^{m-2j} - q^{-n-1} \sum_{j=0}^{m} q^{-m+2j} \right) =$$

$$\sum_{j=0}^{m} (q - q^{-1})^{-1}(q^{n+m-2j+1} - q^{-(n+m-2j+1)}) = \sum_{j=0}^{m} \chi_{V(n+m-2j)}.$$

This proves that if $m < n$ then

$$V(m) \otimes V(n) \cong \bigoplus_{j=0}^{m} V(n + m - 2j) = V(n + m) \oplus V(n + m - 2) \oplus \cdots \oplus V(n - m).$$

Problem 3. Section 8.5 (page 40) #2. For each algebra of type $A_\ell, B_\ell, C_\ell, D_\ell$, determine the roots and root spaces. How are the various h_α expressed in terms of the basis for H given in (1.2)?

Solution. For A_ℓ, we take the maximal toral subalgebra $\mathfrak{h}$ to be the diagonal subalgebra

$$\left\{ \boldsymbol{t} = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{\ell+1} \end{pmatrix} \mid \sum t_i = 0 \right\}.$$

Then $\mathfrak{h}^*$ is generated by the linear functionals $\boldsymbol{e}_i$ such that $\boldsymbol{e}_i(\boldsymbol{t}) = t_i$. It is isomorphic to the quotient of $\mathbb{C}^{\ell+1}$ by the one-dimensional subspace spanned by the vector $(1, 1, \cdots, 1)$ since the vector $\sum \boldsymbol{e}_i$ is zero.

As I have explained, I prefer to use slightly different realizations of the classical Lie algebras than Humphreys. There is no substantial difference since the following realizations will be conjugate to those used by Humphreys in $\mathrm{GL}(N, \mathbb{C})$ where $N = 2\ell + 1$ for B_ℓ and $N = 2\ell$ for C_ℓ and D_ℓ. Let J_N be the $N \times N$ matrix

$$J_N = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Then we define $\mathfrak{so}(N)$, which may also be denoted $\mathfrak{o}(N)$ to be the set of $N \times N$ matrices $X \in \mathrm{Mat}_N(\mathbb{C})$ that satisfy $XJ = -J \cdot {}^t X$ with $J = J_N$.

We define $\mathfrak{sp}(2\ell)$ (Type C_ℓ), For $\mathfrak{so}_{2\ell+1}$ (Type B_ℓ) and $\mathfrak{so}_{2\ell}$ (Type D_ℓ) to be the set of matrices X that satisfy

$$XJ + JX^t = 0$$

where J is as in the following table:

B_ℓ	$J_{2\ell+1}$
C_ℓ	$\begin{pmatrix} & -J_\ell \\ J_\ell & \end{pmatrix}$
D_ℓ	$J_{2\ell}$

Let us define the Cartan subalgebra $\mathfrak{h}$ to be

$$\left\{ \boldsymbol{t} = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_\ell & & & \\ & & & -t_\ell & & \\ & & & & \ddots & \\ & & & & & -t_1 \end{pmatrix} \right\} \cong \mathbb{C}^\ell$$

in the cases C_ℓ and D_ℓ, and

$$\left\{ \boldsymbol{t} = \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_\ell & & & & \\ & & & 0 & & & \\ & & & & -t_\ell & & \\ & & & & & \ddots & \\ & & & & & & -t_1 \end{pmatrix} \right\} \cong \mathbb{C}^\ell$$

in the case B_ℓ. For Humphreys' realizations we would have the diagonal entries in a different order, more precisely $\mathrm{diag}(t_1, \cdots, t_\ell, -t_1, \cdots, -t_\ell)$ for types C_ℓ and D_ℓ and $\mathrm{diag}(t_1, \cdots, t_\ell, -t_1, \cdots, -t_\ell, 0)$ for type B_ℓ.

We will denote by $\boldsymbol{e}_i \in \mathfrak{h}^*$ the linear functional that maps $\boldsymbol{t}$ to t_i. The root systems are:

A_ℓ	$\{\boldsymbol{e}_i - \boldsymbol{e}_j, 1 \leqslant i, j \leqslant \ell + 1, i \neq j\}$
B_ℓ	$\{\pm\boldsymbol{e}_i \pm \boldsymbol{e}_j, 1 \leqslant i, j \leqslant \ell, i \neq j\} \cup \{\boldsymbol{e}_i\}$
C_ℓ	$\{\pm\boldsymbol{e}_i \pm \boldsymbol{e}_j, 1 \leqslant i, j \leqslant \ell, i \neq j\} \cup \{2\boldsymbol{e}_i\}$
D_ℓ	$\{\pm\boldsymbol{e}_i \pm \boldsymbol{e}_j, 1 \leqslant i, j \leqslant \ell, i \neq j\}$

For the h_α here they are for the simple roots in types B_3 and C_3, and one non-simple root.

Type B_3

The simple roots are $\alpha_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$, $\alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3$ and $\alpha_3 = \boldsymbol{e}_3$. We list the h_α and x_α; the $x_{-\alpha}$ are the transposes of the x_α (except for α_3, where it is twice the transpose). We also do one non-simple root $\boldsymbol{e}_1 + \boldsymbol{e}_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3$

α	h_α	x_α
$\alpha_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$	$\begin{pmatrix} 1 &&&&&& \\ & -1 &&&&& \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 && \\ &&&&& 1 & \\ &&&&&& -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 &&&&& \\ & 0 &&&&& \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 && \\ &&&&& 0 & -1 \\ &&&&&& 0 \end{pmatrix}$
$\alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3$	$\begin{pmatrix} 0 &&&&&& \\ & 1 &&&&& \\ && -1 &&&& \\ &&& 0 &&& \\ &&&& 1 && \\ &&&&& -1 & \\ &&&&&& 0 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&&& \\ & 0 & 1 &&&& \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 & -1 & \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}$
$\alpha_3 = \boldsymbol{e}_3$	$\begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 2 &&&& \\ &&& 0 &&& \\ &&&& -2 && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 0 & 1 &&& \\ &&& 0 & -1 && \\ &&&& 0 && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}$
$\boldsymbol{e}_1 + \boldsymbol{e}_2$	$\begin{pmatrix} 1 &&&&& 1 & \\ & 1 &&&&& -1 \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 && \\ &&&&& -1 & \\ &&&&&& -1 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& 1 & \\ & 0 &&&&& -1 \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}$

Type C_3

The simple roots are $\alpha_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$, $\alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3$ and $\alpha_3 = 2\boldsymbol{e}_3$. We list the h_α and x_α; the $x_{-\alpha}$ are the transposes of the x_α. We also do one non-simple root $\boldsymbol{e}_1 + \boldsymbol{e}_2 = \alpha_1 + 2\alpha_2 + \alpha_3$

α	h_α	x_α
$\alpha_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$	$\begin{pmatrix} 1 &&&&& \\ & -1 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 1 & \\ &&&&& -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 &&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & -1 \\ &&&&& 0 \end{pmatrix}$
$\alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3$	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && -1 &&& \\ &&& 1 && \\ &&&& -1 & \\ &&&&& 0 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& \\ & 0 & 1 &&& \\ && 0 &&& \\ &&& 0 & -1 & \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}$
$\alpha_3 = \boldsymbol{e}_3$	$\begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& -1 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 & 1 && \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}$
$\boldsymbol{e}_1 + \boldsymbol{e}_2$	$\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& -1 & \\ &&&&& -1 \end{pmatrix}$	$\begin{pmatrix} 0 &&&& 1 & \\ & 0 &&&& 1 \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}$

Problem 4. Section 8.5 (page 40) #8. (Do $\mathfrak{sl}_3$ only.) Calcuate explicitly the root strings and Cartan integers. In particular prove that all Cartan integers $2(\alpha|\beta)/(\beta|\beta|)$ with $\alpha \neq \beta$ for $\mathfrak{sl}_n$ are $0, \pm 1$.

Solution. (Omitted.)

Problem 5. Section 8.5 (page 40) #10. Prove that no four, five or seven dimensional semisimple Lie algebras exist.

Solution. Let $\dim(\mathfrak{h}) = \ell$. We have $\dim(\mathfrak{g}) = \dim(\mathfrak{h}) + |\Phi|$, where Φ is the root system. The number of simple roots equals ℓ, and for each simple root, there is also its negative, so $|\Phi| \geqslant 2\ell$ and thus $\dim(\mathfrak{g}) \geqslant 3\ell$. So if $\dim(\mathfrak{g}) \leqslant 9$, we need only consider the cases $\ell = 1$ or 2.

We know the possible root systems when $\dim(\mathfrak{h}) = 1$ (A_1 only, with $\dim(\mathfrak{g}) = 3$), and when $\dim(\mathfrak{h}) = 2$:

Φ	$\lvert\Phi\rvert$	$\dim(\mathfrak{g})$
$A_1 \times A_1$	2	6
A_2	6	8
$B_2 = C_2$	8	10
G_2	12	14

We see that there are semisimple Lie algebras of dimensions $3, 6, 8$ but none of dimensions 4, 5 or 7.

Problem 5$'$. Section 8.5 (page 40) #11. If $(\alpha, \beta) > 0$, prove that $\alpha - \beta \in \Phi$ for $\alpha, \beta \in \Phi$. Is the converse true?

Solution. In Section 8.5, Humphreys defines the inner product (α,β) to equal $\kappa(t_\alpha, t_\beta)$. Let us show that the express the Cartan integer $\beta(h_\alpha)$, which appears in Section 8.4 in terms of the inner product. We will prove

$$\boxed{\beta(h_\alpha) = \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)}.}$$

Indeed

$$\beta(h_\alpha) = \beta\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right) = 2\frac{\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Now let q and r be as in Proposition 8.4 (e). Then $r - q = \beta(h_\alpha) > 0$ and since r, q are nonnegative integers, $r \geqslant 1$. Thus $-r \leqslant -1 \leqslant q$ and so by Proposition 8.4 (e), $\beta - \alpha$ is a root. Thus its negative $\alpha - \beta$ is a root.

The converse is not true and we may use the G_2 root system to give a counterexample. Let α_1, α_2 be the short and long simple root (labeled α and β in Figure 1 on page 44). Now let $\alpha = \alpha_1$ and $\beta = \alpha_1 + \alpha_2$. Then $\alpha - \beta$ is a root, but $(\alpha, \beta) < 0$.