Math 210C Homework 4

- Section 7 (p.34) # 2,6,7,
- Section 8 (p.40) # 5,8,11.

Section 7 #2. Let $M = \mathfrak{sl}(3, F)$. Then M contains a copy of $L = \mathfrak{sl}(2, F)$ in its upper left-hand corner. Write M as a direct sum of irreducible L-submodules (M viewed as an L-module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Section 7 #6. Decompose the tensor product of the $L = \mathfrak{sl}(2)$ modules $V(3)$, $V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

In the next exercise I am changing notation and writing $M(\lambda)$ instead of $Z(\lambda)$. The notation has become standardized in the years since Humphrey's book was written, and $M(\lambda)$ is nowadays called a *Verma module*. Humphreys calls $M(\lambda)$ a *standard cyclic module* and denotes it $Z(\lambda)$. Verma modules become important later in the book, where following BGG he uses these infinite dimensional modules to study the finite-dimensional irreducibles. I am not asking you to do (c), though (c) is not hard if you do the last part of (a).

Section 7 #7. In this exercise we construct certain *infinite-dimensional* L-modules. Let $\lambda \in F$ be an arbitrary scalar. Let $M(\lambda)$ be a vector space over F with countable basis $(v_0, v_1, v_2, \cdots)$.

(a) Prove that formulas (a)-(c) of Lemma (7.2) define an L-module structure on $M(\lambda)$, and that every nonzero L-submodule of $M(\lambda)$ contains at least one maximal vector.

(b) Suppose $\lambda + 1 = i$ is a positive integer. Prove that this induces an L-module homomorphism $M(\mu) \xrightarrow{\phi} M(\lambda)$, $\mu = \lambda - 2i$, sending v_0 in $M(\mu)$ to v_i in $M(\lambda)$.

Section 8 #5. If L is semisimple, H a maximal toral subalgebra, prove that H is self-normalizing (i.e $H = N_L(H)$).

Section 8 #8. For $\mathfrak{sl}(n, F)$ calculate the root strings and Cartan integers. In particular prove that all Cartan integers $2(\alpha, \beta)/(\beta, \beta)$ with $\alpha \neq \beta$ for $\mathfrak{sl}(n)$ are $0, 1, -1$.

Section 8 #11. If $(\alpha, \beta) > 0$, prove that $\alpha - \beta \in \Phi$ $(\alpha, \beta \in \Phi)$. Is the converse true?