Homework 3 Solutions

Problem 1: Section 4.3 (page 20) #7. Prove the converse to Theorem 4.3 (Cartan's Criterion).

Solution: We show that if $L \subseteq \mathfrak{gl}(V)$ is solvable then $\mathrm{tr}(xy) = 0$ for $x \in [L, L]$, $y \in L$. By Lie's theorem, L stabilizes a flag

$$V = V_n \supset V_{n-1} \supset \cdots \supset V_0 = 0, \qquad \dim(V_i) = i.$$

Choose a basis of V so that V_k is the span of $v_1, \cdots, v_k$. With respect to this basis, any element x of V is upper triangular, so

$$x = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ & \lambda_2 & & * \\ & & \ddots & \vdots \\ & & & \lambda_n \end{pmatrix}.$$

Moreover an element $y \in [L, L]$ is a commutator of upper triangular matrices, so it is upper triangular and nilpotent:

$$y = \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & & * \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}.$$

Therefore xy is also upper triangular and nilpotent so $\mathrm{tr}(xy) = 0$.

Problem 2: Section 5.4 (page24) #1. Prove that if L is nilpotent, the Killing form of L is identically zero.

Solution. Let $L = L^0 \supset L^1 \supset \cdots \supset L^n = 0$ be the descending series, so if $x \in L$ then $\mathrm{ad}(x)L^i \subseteq L^{i+1}$. Thus $\mathrm{ad}(x)\,\mathrm{ad}(y)L^i \subseteq L^{i+2}$ and so $(\mathrm{ad}(x)\,\mathrm{ad}(y))^{n-1} = 0$. We see that $\mathrm{ad}(x)\,\mathrm{ad}(y)$ is nilpotent so $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$.

Problem 3: Section 5.4 (page 24) #5. Let $L = \mathfrak{sl}(2, F)$. Compute the basis of L with respect to the standard basis, relevant to the Killing form.

Solution. We use the basis $\{H, E, F\}$ where

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

With respect to this basis

$$\operatorname{ad}(H) = \begin{pmatrix} 0 & & \\ & 2 & \\ & & -2 \end{pmatrix}, \qquad \operatorname{ad}(E) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{ad}(F) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

From this, we compute that the Killing form is given by the following table:

Thus the dual basis to $\{H, E, F\}$ is $\left\{\frac{1}{8}H, \frac{1}{4}F, \frac{1}{4}E\right\}$.

Problem 4: Section 6.4 (page 30) #1. Using the standard basis for $L = \mathfrak{sl}(2, F)$, write down the Casimir element of the adjoint representation of L (cf. Exercise 5.5). Do the same thing for the usual (3-dimensional) representation of $\mathfrak{sl}(3, F)$, first computing dual bases relative to the trace form.

Solution: For $\mathfrak{sl}_2$, using the result of the previous problem, the Casimir element of the adjoint representation is

$$\operatorname{ad}(H) \operatorname{ad}\left(\frac{1}{8}H\right) + \operatorname{ad}(E) \operatorname{ad}\left(\frac{1}{4}F\right) + \operatorname{ad}(F) \operatorname{ad}\left(\frac{1}{4}\right) E.$$

In terms of the matrices for ad computed in the previous problem, this equals

$$\frac{1}{8}\begin{pmatrix} 0 & & \\ & 2 & \\ & & -2 \end{pmatrix}^2 + \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which works out to

$$\frac{1}{8}\begin{pmatrix} 0 & & \\ & 4 & \\ & & 4 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 & & \\ & 2 & \\ & & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 & & \\ & 0 & \\ & & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

This is consistent with the computation at the bottom of page 27 that $c_\phi = \dim(L)/\dim(V)$ which equals 1 when $V = L$ (for the adjoint representation).

For the standard representation of $\mathfrak{sl}_3$, dual bases of $\mathfrak{g}$ for the trace bilinear form in the standard representation may be taken to be

x_i	y_i	x_iy_i
$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix}$	$\begin{pmatrix} 2/3 & & \\ & 1/3 & \\ & & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 1/3 & \\ & & 2/3 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$

Thus

$$\sum x_iy_i = \begin{pmatrix} 8/3 & & \\ & 8/3 & \\ & & 8/3 \end{pmatrix},$$

which is again consistent with the formula $c_\phi = \dim(L)/\dim(V)$, with $\dim(L) = 8$ and $\dim(V) = 3$.

Problem 5: Section 6.4 (page 30) #6. Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two associate bilinear forms on L. If β and γ are nondegenerate, prove that β and γ are proportional. [Use Schur's Lemma.]

Solution. If V is any L-module, we may make the dual space V^* into a module (the dual or contragredient module) by

$$(x\lambda)(v) = -\lambda(xv) \tag{1}$$

for $x \in L$, $\lambda \in V^*$, $v \in V$. To check that this is a representation, we must show that

$$[x, y]\lambda = x(y\lambda) - y(x\lambda). \tag{2}$$

Indeed,

$$\begin{aligned}([x, y]\lambda)(v) &= -\lambda([x, y]v) = -\lambda(xy(v)) + \lambda(yx(v)) \\ &= -(x\lambda)(yv) + (y\lambda)(xv) = -y(x\lambda)(v) + x(y\lambda)(v),\end{aligned}$$

proving (2). Now let β be an associative form, and define $\phi_\beta : L \longrightarrow L^*$ by

$$\phi_\beta(x)(y) = \beta(x, y).$$

Lemma 1. *ϕ_β is a L-module homomorphism.*

Proof. Note that if $\lambda \in L^*$ then the contragredient action (1) means that

$$(x\lambda)(z) = -\lambda(\mathrm{ad}(x)z) = -\lambda([x, z]), \qquad x, z \in \mathfrak{g}, \lambda \in \mathfrak{g}^*.$$

So to check that $\phi_\beta(\mathrm{ad}(x)y) = (x\phi_\beta)(y)$ we need

$$\phi_\beta(\mathrm{ad}(x)y)(z) = -\phi_\beta(y)([x, z]),$$

or

$$\beta([x, y], z) = -\beta(y, [x, z]).$$

This is equivalent to the associativity of the form β. □

Now since L is simple, the adjoint representation is irreducible. It follows easily that the module L^* is also irreducible. By Schur's Lemma, the space of L-module homomorphisms $L \to L^*$ is at most one-dimensional, so ϕ_β and ϕ_γ are proportional. Therefore β and γ are also proportional.

Problem 6: Section 6.4 (page 30) #7. It will be seen later that $\mathfrak{sl}(n, F)$ is actually simple. Assuming this and using Exercise 6, prove that the Killing form κ on $\mathfrak{sl}(n, F)$ is related to the ordinary trace form by $\kappa(x, y) = 2n\,\mathrm{tr}(xy)$.

Solution. Both forms are associative therefore (by the last exercise) proportional, so it is sufficient to prove $\kappa(x, y) = 2n\,\mathrm{tr}(xy)$ for any x, y such that $\mathrm{tr}(xy) \neq 0$. We choose $x = y = H$ where

$$H = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

We will show that

$$\kappa(H, H) = 4n, \qquad \mathrm{tr}(H^2) = 2.$$

The second identity is clear. For the first, we need to compute the eigenvalues of $\mathrm{ad}(H)^2$. We will denote by E_{ij} the elementary matrix with 1 in the i, j position, 0 elsewhere. Any element of the diagonal Cartan subalgebra is an eigenvector of $\mathrm{ad}(H)$ with eigenvalues 0. The remaining eigenvectors are each of the form E_{ij}. These also have eigenvalue 0 if $i > 2$ and $j > 2$. The remaining cases are given by the following table.

vector	E_{12}	E_{21}	$E_{1j}, \quad j > 2$	$E_{2j}, \quad j > 2$	$E_{i1}, \quad i > 2$	$E_{j2}, \quad i > 2$
eigenvalue	2	-2	1	-1	-1	1

Then $\mathrm{tr}(\mathrm{ad}(H)^2)$ is the sum of the squares of these eigenvalues, that is

$$1 \cdot 2^2 + 1 \cdot (-2)^2 + (n-2) \cdot 1^2 + (n-2) \cdot (-1)^2 + (n-2) \cdot (-1)^2 + (n-2) \cdot 1^2$$

which equals $4n$.