## Math 210C Homework 2 Solutions

Section 2.3 \#1. Prove that the set of all inner derivations $\operatorname{ad}(x), x \in L$ is an ideal of $\operatorname{Der}(L)$. What is the group analog of this result?

Solution. Suppose that $D \in \mathfrak{g l}(L)$ is a derivation of $L$, and that $\operatorname{ad}(x)$ is an inner derivation (for some $x \in L$ ). We show that

$$
\begin{equation*}
[D, \operatorname{ad}(x)]=\operatorname{ad}(D(x)) \tag{1}
\end{equation*}
$$

Note that this implies that the subalgebra of $\operatorname{Der}(L)$ consisting of inner derivations is an ideal, as required. Indeed, applying $[D, \operatorname{ad}(x)]$ to $y \in L$ gives

$$
D([x, y])-[x, D y]=[D x, y]+[x, D y]-[x, D y]=\operatorname{ad}(D x) y
$$

proving (1) and we are done.
The corresponding result for groups is that the subgroup of the automorphism group $\operatorname{Aut}(G)$ of a group $G$ consisting of inner automorphisms is a normal subgroup.

Section $2.3 \# 7$. Prove that $\mathfrak{t}(n, F)$ and $\mathfrak{d}(n, F)$ are self-normalizing subalgebras of $\mathfrak{g l}(n, F)$, whereas $\mathfrak{n}(n, F)$ has normalizer $\mathfrak{t}(n, F)$.

Solution. I will use the notation $E_{i j}$ to represent the matrix with 1 in the $i, j$ position, 0 elsewhere. Suppose that $x$ normalizes $\mathfrak{t}=\mathfrak{t}(n, F)$. This means that $[x, y]$ is upper triangular whenever $y$ is upper triangular. If $x$ is not in $\mathfrak{t}$ then $x_{i j} \neq 0$ for some $i>j$. Let $y=E_{j j} \in \mathfrak{t}$. Then

$$
[x, y]_{i j}=x_{i j} \neq 0
$$

so $[x, y] \notin \mathfrak{t}$. This is a contradiction since $x \in N(\mathfrak{t})$. We have proved that $N(\mathfrak{t})=\mathfrak{t}$.
Next suppose that $x=\left(x_{i j}\right)$ normalizes $\mathfrak{d}=\mathfrak{d}(n, F)$. We must show that $x \in \mathfrak{d}$. If not, $x_{i j} \neq 0$ for some $i \neq j$ and then $[x, y]_{i j}=x_{i j}$ with $y=E_{j j} \in \mathfrak{d}$, contradiction since $x \in N(\mathfrak{d})$.

Finally, we must show that $N(\mathfrak{n})=\mathfrak{t}$. If $x \in \mathfrak{n}$ and $y \in \mathfrak{t}$ then $x y$ and $y x$ are upper triangular and nilpotent, hence so is $[x, y]=x y-y x$. This proves $\mathfrak{t} \subseteq N(\mathfrak{n})$. Conversely suppose that $x \notin \mathfrak{t}$ so $x_{i j} \neq 0$ for some $i>j$. Now $y=E_{j i} \in \mathfrak{n}$ but $[x, y]_{j j}=x_{i j}$. So $[x, y] \notin \mathfrak{n}$ and thus $x \notin N(\mathfrak{n})$.

Section 3.3\#1. Let $I$ be an ideal of $L$. Then each member of the derived series or descending central series of $I$ is also an ideal of $L$.

The following basic fact (mentioned in Humphreys, page 6) implies the exercise since the ideals in the derived series and descending central series are obtained from $L$ by successive bracketing.

Lemma 1. If $I, J$ are ideals in $L$, then so is $[I, J]$.
Proof. If $x \in I, y \in J$ and $z \in L$ we need to show that $[z,[x, y]] \in[I, J]$. Indeed by the Jacobi identity,

$$
[z,[x, y]]=[[z, x], y]+[x,[z, y]] \in I+J
$$

since $I$ and $J$ are ideals.

Section 3.3 \#2. Prove that $L$ is solvable if and only if there exists a chain of subalgebras $L=L_{0} \supset L_{1} \supset L_{2} \supset \ldots \supset L_{k}=0$ such that $L_{i+1}$ is an ideal of $L_{i}$ and such that each quotient $L_{i} / L_{i+1}$ is abelian.

Solution. We start with:
Lemma 2. If $L$ is a Lie algebra and $I$ is an ideal then $L / I$ is abelian if and only if $I \supseteq[L, L]$.
Proof. If $x \in L$ let $\bar{x}$ denote the image of $x$ in $L / I$. Since $[\bar{x}, \bar{y}]=\overline{[x, y]}$ it is clear that $[\bar{x}, \bar{y}]=0$ for all $\bar{x}, \bar{y} \in L$ if and only if $I \supseteq[L, L]$.

By the Lemma, if $L$ is solvable and $L^{(i)}$ is the derived series, we may take $L_{i}=L^{(i)}$ and obtain a chain of subalgebras $L_{0} \supset L_{1} \supset \cdots$ such that $L_{i+1}$ is an ideal of $L_{i}$ with abelian quotients. Conversely, suppose we are given such a chain. Since $L_{0} / L_{1}$ is abelian, by the Lemma, $L_{1} \supseteq[L, L]=L^{(1)}$. Then since $L_{1} / L_{2}$ is abelian, $L_{2} \supseteq\left[L_{1} / L_{1}\right] \supseteq\left[L^{(1)}, L^{(1)}\right]=L^{(2)}$. Continuing this way we eventually get $L_{n} \supseteq L^{(n)}$. But $L_{n}=0$ so $L^{(n)}=0$. Therefore $L$ is solvable.

Section 3.3 \# 5. Prove that the nonabelian two dimensional algebra constructed in (1.4) is solvable but not nilpotent. Do the same for the algebra in Exercise 1.2.

Solution. The first Lie algebra is $F x \oplus F y$ where $[x, y]=x$. The derived series is $L^{(1)}=F x$ and $L^{(2)}=[F x, F x]=0$, so this Lie algebra is solvable. However the descending central series has $L^{1}=F x$ and $L^{2}=[L, F x]=F x$ so $L^{n}=F x$ for all $n \geqslant 1$. Thus the Lie algebra is not nilpotent.

For the second Lie algebra, $L=F x \oplus F y \oplus F z$ where $[x, y]=z,[x, z]=y$ and $[y, z]=0$. In this case we find $L^{(1)}=F y \oplus F z$ and $L^{(2)}=0$, but $L^{n}=F y \oplus F z$ for all $n \geqslant 1$. So this Lie algebra also is solvable but not nilpotent.

Section 3.3 \# 6. Prove that the sum of two nilpotent ideals of a Lie algebra $L$ is again a nilpotent ideal. Therefore $L$ posesses a unique maximal nilpotent ideal. Determine this ideal for each algebra in Exercise 5.

Solution. Suppose that $I$ and $J$ are nilpotent ideals, and let $K=I+J$. Let $I^{i}, J^{i}, K^{k}$ be the descending central series. Let $K=I+J$. We will show that

$$
\begin{equation*}
K^{n} \subseteq K_{n} \tag{2}
\end{equation*}
$$

where we define

$$
K_{n}=I^{n}+J^{n}+\sum_{i=0}^{n-1} I^{i} \cap J^{n-1-i}
$$

The sum is by definition zero if $n=0$, so there are no terms, and $K_{0}=I^{0}+J^{0}=I+J=K$. Thus (2) is true if $n=0$. Arguing by induction, assume (2) and note that

$$
\left[I, K_{n}\right] \subseteq\left[I, I^{n}\right]+\left[I, J^{n}\right]+\sum_{i=0}^{n-1}\left[I, I^{i} \cap J^{n-1-i}\right] \subseteq I^{n+1}+I \cap J^{n}+\sum_{i=0}^{n-1} I^{i+1} \cap J^{n-1-i}
$$

whence $\left[I, K_{n}\right] \subseteq K_{n+1}$ and similarly $\left[J, K_{n}\right] \subseteq K_{n+1}$. Therefore

$$
K^{n+1}=\left[I, K^{n}\right]+\left[J, K^{n}\right] \subseteq K_{n+1} .
$$

This proves (2).
Since $I$ and $J$ are nilpotent, $K_{n}=0$ for $n$ sufficiently large proving that $K^{n}=0$ and therefore $K$ is nilpotent. We have proved that the sum of two nilpotent ideals is nilpotent. Thus if $K$ is a maximal nilpotent ideal then $K$ contains every nilpotent ideal $I$ (since otherwise $I+K$ is a strictly larger nilpotent ideal). Therefore $L$ has a unique maximal nilpotent ideal.

Remark 1. The next two problems were described as being on page 20 and in Section 5.1. This was an error on my part. The two problems on page 20 are in Section 4.3. (There are no problems in Section 5.1.) If you did two different problems because of my mistake, I will accept those instead.

Remark 2. Problem 4.3 \#1 is different in earlier editions of the book. The following version is in the 1994 edition. In my old 1972 version of the book, the same question is asked for the classical Lie algebras, not just $\mathfrak{s l}(V)$.

Section $4.3 \# 1$. Let $L=\mathfrak{s l}(V)$. Use Lie's Theorem to prove that $\operatorname{Rad}(L)=Z(L)$; conclude that $L$ is semisimple (cf. Exercise 2.3). [Observe that $\operatorname{Rad}(L)$ lies in each maximal subalgebra $B$ of $L$. Select a basis of $V$ so that $B=L \cap \mathfrak{t}(n, F)$, and notice that the transpose of $B$ is also a maximal solvable subalgebra of $L$. Conclude that $\operatorname{Rad}(L) \subset L \cap \mathfrak{d}(n, F)$, then that $\operatorname{Rad}(L)=Z(L)$.]

Solution. We must prove
Lemma 3. Let $L$ be a finite-dimensional Lie algebra. If $B$ is a maximal solvable subalgebra then $\operatorname{rad}(L) \subseteq B$.

Proof. We recall that if $M^{\prime}$ is a solvable ideal in the Lie algebra $M$ and if $M / M^{\prime}$ is also solvable then $M$ is solvable. To apply this take $M=B+\operatorname{rad}(L)$ and $M^{\prime}=\operatorname{rad}(L)$. Clearly $M^{\prime}$ is a solvable ideal in $M$ and $M / M^{\prime} \cong B /(B \cap \operatorname{rad}(L))$ is solvable since it is a homomorphic image of the solvable Lie algebra $B$. Therefore $B+\operatorname{rad}(L)$ is solvable and by the assumed maximality of $B$ we have $B+\operatorname{rad}(L)=B$, whence $\operatorname{rad}(L) \subseteq B$.

Now to solve the problem, identify $L=\mathfrak{s l}(V)=\mathfrak{s l}(n, F)$. By Lie's theorem, every solvable Lie subalgebra of $\mathfrak{s l}(V)$ stabilizes a flag. Hence the stabilizer of a flag is maximal solvable, and therefore contains $\operatorname{rad}(L)$. Both $\mathfrak{t}$ and its transpose $\mathfrak{t}^{t}$ are stabilizers of flags, $\operatorname{so} \operatorname{rad}(V) \subseteq$ $\mathfrak{t} \cap \mathfrak{t}^{t}=\mathfrak{d}$. Now let $x \in \operatorname{rad}(V)$. Since we know $x$ is diagonal, assume $x=\operatorname{diag}\left(x_{1}, \cdots, x_{n}\right)$. If two of the eigenvalues $x_{i}$ and $x_{j}$ are distinct, then $\left[x, E_{i j}\right]=\left(x_{i}-x_{j}\right) E_{i j}$ is not diagonal, so $\left[x, E_{i j}\right] \notin \mathfrak{d}$. This is a contradiction since $\operatorname{rad}(V)$ is an ideal. Thus $x$ is a scalar linear transformation, hence $x \in Z(L)$. This proves that $\operatorname{rad}(V) \subseteq Z(L)$. Then remembering that $L=\mathfrak{s l}(n, F), Z(L)=0$ so $L$ is semisimple.

Section $4.3 \# 5$. If $x, y \in \operatorname{End}(V)$ commute, prove that $(x+y)_{s}=x_{s}+y_{s}$ and $(x+y)_{n}=$ $x_{n}+y_{n}$. Show by example that this can fail if $x, y$ fail to commute. [Show first that $x, y$ semisimple (resp. nilpotent) implies $x+y$ semisimple (resp. nilpotent).]
Lemma 4. (i) Let $x$ and $y$ be commuting semisimple endomorphisms. Then $x+y$ is semisimple.
(ii) Let $x$ and $y$ be commuting nilpotent endomorphisms. Then $x+y$ is nilpotent.

Proof. For (i), observe that each $x$ eigenspace is $y$ invariant, since if $x v=\lambda v$ then $x y v=$ $y x v=\lambda y v$. Because $x$ and $y$ are both semisimple, $V$ is the direct sum of $x$-eigenspaces, each of which is $y$ invariant, and the endomorphisms on these $x$-eigenspaces induced by $y$ are semisimple since $y$ is semisimple. Thus $x$ and $y$ can be simultaneously diagonalized. It follows that $x+y$ is semisimple.

For (ii), if $x^{n}=0$ and $y^{m}=0$, and $x y=y x$, then $(x+y)^{n+m}=\sum\binom{n}{k} x^{n+m-k} y^{k}=0$, so $x+y$ is nilpotent.

Now note that since $x$ and $y$ commute, and since by the Jordan decomposition $x_{s}$ and $x_{n}$ are polynomials in $x$ and $y_{s}$ and $y_{n}$ are polynomials in $y$, all elements $x, y, x_{s}, x_{n}, y_{s}, y_{n}$ all commute. By the Lemma, $x_{s}+y_{s}$ is semisimple and $x_{n}+y_{n}$ is nilpotent. By the uniqueness assertion of the Jordan decomposition (Proposition 4.2) it follows that $(x+y)_{s}=x_{s}+y_{s}$ and $(x+y)_{n}=x_{n}+y_{n}$.

We are also asked to give counterexamples when $x$ and $y$ do not commute. Here are two examples. In the first, $x, y$ are semisimple but $x+y$ is nilpotent (and not semisimple):

$$
x=\left(\begin{array}{cc}
1 & 1 \\
& -1
\end{array}\right), \quad y=\left(\begin{array}{cc}
-1 & \\
& 1
\end{array}\right), \quad x+y=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) .
$$

In the second example, $x, y$ are nilpotent but $x+y$ is semisimple (and not nilpotent):

$$
x=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right), \quad y=\left(\begin{array}{cc}
0 & 0 \\
-1 & 0
\end{array}\right), \quad x+y=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) .
$$

