Frobenius Groups (III)

Daniel Bump

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Introduction

In our last lectures we gave Frobenius' theorem as an example of how character theory can be used to prove theorems in group theory that, on the face of it, don't seem to involve representations. Yet Frobenius' theorem has never been proved without representation theory.

Moreover Frobenius' theorem (proved last time) is an instructive one because it is quite instructive on the interaction between characters, conjugacy classes and how the representation theory reflects the subgroup structure is related to the representations of the group.

Today we will take this analysis further and describe all irreducible representations of a Frobenius group.

Extending versus inducing

One theme for today's lecture is that there is a parallel between representations and conjugacy classes.

Another theme is that if G is a finite group and H a subgroup, there are two ways that we can go from an irreducible representation π of H to a representation of G.

- We may try to extend π to a representation of G, or
- We may induce π to a representation of G.

We are interested in getting an irreducible.

A test case

If $[G:H]=2$, then H is normal in G. There are exactly two things that can happen.

- It is possible that π can be extended to a representation of G. If this is true, there are two ways to do it.
- If π cannot be extended, then its induction is irreducble.

The proof (using Frobenius reciprocity) is not very difficult. But this result is worth thinking about because it is a very common situation.

There is a similar dichotomy for conjugacy classes when $[G:H]=2$. Today we will review a similar dichotomy for representations of Frobenius groups, in which the two methods (extension or induction) appear as alternatives. And we will see how information about conjugacy classes gives information about representations.

Review: Frobenius groups

Recall that a Frobenius group is a group G that acts transitively on a set X such that no non-identity element fixes more than one point. The isotropy group H of a point is called a Frobenius complement. Frobenius's group says that G is a semidirect product, $G = HK$ where K is a normal subgroup. Moreover $K = K^* \cup \{1\}$ where K^* is the set of elements that fix no elements of X.

The proof involved showing that every irreducible representation of H can be extended to an irreducible representation of G. Today we will prove the complementary result that every irreducible character of K (except the trivial character) can be induced to an irreducible of K. The combination of the two will give us all characters.

Dummit and Foote definition

Last week we considered the relationship between the Dummit and Foote definition of a Frobenius group and the usual one that we've adopted.

- Standard definition: A Frobenius group is a permutation group in which no element but the identity fixes more than one point;
- Dummit and Foote definition: A group with a normal subgroup K such that if $1 \neq k \in K$ then the centralizer $C(k) \subseteq K$.

We proved that their definition implies that G is a Frobenius group.

Today let us check the converse.

A Frobenius group satisfies the Dummit and Foote definition

Proposition

Let G be a Frobenius group acting on the set X with Frobenius kernel K. Let $1 \neq k \in K$. Then $C(k) \subseteq K$.

Suppose that $h \notin K$ commutes with k. Then h has a fixed point $x \in X$. Thus $h = khk^{-1}$ also has the fixed point kx. Because no nonidentity element of G has two fixed points, $x = kx$ which is a contradiction since k has no fixed points.

H acts freely on K^* so $|H| \mid |K| - 1$

In our previous lecture on this, we observed a property of K, that it is a Normal Hall subgroup. The Hall property means that $(|K|, [G : K]) = 1$ or equivalently $(|K|, |H|) = 1$. We can now improve this result.

Proposition

Let G be a Frobenius group with Frobenius kernel K and complement H. Then $|H|$ divides $|K| - 1$.

Let H act on the set K^* of nonidentity elements of K. If $1 \neq h \in H$ then h cannot have a fixed point in K^* since if $hkh^{-1} = k$ then $h \in C(k)$, contradiction since $C(k) \subseteq K$. Thus each orbit in this action has cardinality $|H|$ and so $|H|$ divides $|K^*| = |K| - 1$.

A Frobenius group with nonabelian Kernel

So far in all our examples, the Frobenius kernel was abelian. We give an example (from Passman: Permutation Groups) to show this is not always necessarily true.

Let p be a prime, and let F be the field $\mathbb{F}_{p^n}$ where $n > 1$. The Frobenius map $\phi : F \longrightarrow F$ defined by $\phi(x) = x^p$ is an automorphism, an element of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. We assume that n is odd, which implies that $\gcd\left(p+1, \frac{p^n-1}{p-1}\right) = 1$ since

$$\frac{p^n - 1}{p - 1} = p^{n-1} + p^{n-2} + \cdots + 1 = 1 + (p+1)(p + p^3 + \cdots + p^{n-2}).$$

A nonabelian Frobenius Kernel

Consider the group K, with matrices in F of elements of the form

$$\begin{pmatrix} 1 & a^p & b \\ & 1 & a \\ & & 1 \end{pmatrix}.$$

We must also describe the Frobenius complement. Let H be the kernel of the norm map $\mathbb{F}_{p^n}^\times \longrightarrow \mathbb{F}_p^\times$. Explicitly the norm is

$$N(h) := h^{1+p+p^2+\ldots+p^{n-1}} = h^{\frac{p^n-1}{p-1}}.$$

The kernel of the norm map is the unique subgroup of the cyclic group $F^\times$ of index $q-1$.

Example, continued

Embed H into $\mathrm{GL}(3, F)$ as follows. Let $h \in H$, so $h \in F$ and $N(h) = 1$. Identify h with the matrix

$$\begin{pmatrix} h^{p+1} & & \\ & h & \\ & & 1 \end{pmatrix}$$

Then H normalizes K.

Note that since $\gcd\left(p+1, \frac{p^n-1}{p-1}\right) = 1$ if $h \neq 1$ then $h^{p+1} \neq 1$ also. Using this, it is easy to see that h has no fixed points in K if $1 \neq h \in H$. This implies that the semidirect product $H \ltimes K$ is a Frobenius group.

Thompson’s theorem

Although Frobenius kernels can be nonabelian, they are still highly constrained. John Thompson in his thesis (1959) proved that they are nilpotent, a mild generalization of abelian. A group is nilpotent if it is the direct product of its Sylow subgroups.

The example shows a nonabelian Frobenius kernel. Zassenhaus (1935) showed that $\mathrm{SL}(2, \mathbb{F}_5)$ can be a Frobenius complement, but this is essentially the only nonsolvable Frobenius complement.

Action of H on conjugacy classes of K

We have considered the action of H on K by conjugation, and we saw that $1 \neq h \in H$ has only one fixed point (the identity). We may similarly consider the action of H on conjugacy classes of K, and we will see that again, $1 \neq h \in H$ only fixes one conjugacy class of K, namely the identity class. We will leverage this information to obtain information about characters.

Lemma

Let G be a Frobenius group with kernel K and complement H. Let $\mathcal{C}$ be a nonidentity conjugacy class of K. If $h \in H$ and $h\mathcal{C}h^{-1} = \mathcal{C}$ then $h = 1$.

Suppose $h\mathcal{C}h^{-1} = \mathcal{C}$. If $k \in K$ this means that hkh^{-1} is conjugate to k in K, so $hkh^{-1} = \kappa k \kappa^{-1}$ for some $\kappa \in K$. Then $\kappa^{-1}h$ commutes with k. Since $C(k) \subseteq K$ we have $\kappa^{-1}h \in K$ so $h \in H \cap K$ and $h = 1$ which contradicts our assumption.

Our goal today

If τ is a character of K and $h \in H$ let ${}^h\tau$ be the character ${}^h\tau(k) = \tau(h^{-1}kh)$. Our goal today is to prove:

Theorem

Let G be a Frobenius group and let τ be a nontrivial irreducible character of the Frobenius kernel K. Then τ^G is an irreducible representation of G. If τ' is another irreducible character of K then $\tau^G = (\tau')^G$ if and only if $\tau' = {}^h\tau$. The $|H|$ characters ${}^h\tau$ are all distinct, so induction gives an $|H|$-to-one map from irreducible nontrivial characters of K to characters of G.

Thus there are complementary constructions for irreducibles of the Frobenius group G.

- Extend irreducible characters from H to G;
- Induce irreducible characters from K to G.

Richard Brauer

The proof will also illustrate a clever idea of Brauer who, after Frobenius, Schur and Burnside, is the most important historical figure in the representation theory of finite groups. While Frobenius, Schur and Burnside were an earlier generation, Brauer lived 1901-1977. During his years in Germany, he did important work on central simple algebras in the 1930's. In 1933 he was forced to leave Europe by the Nazis and eventually settled in Toronto. His greatest accomplishment was the development of modular representation theory into a powerful tool, and he proved many important theorems even in his later years.

- Richard Brauer (Wikipedia)

Brauer's Lemma on character tables

Lemma (Brauer)

Let G be a group with irreducible characters $\{\chi_1, \cdots, \chi_h\}$ and conjugacy classes $\mathcal{C}_1, \cdots, \mathcal{C}_h$. Let A be another group with actions on both the irreducible characters and the conjugacy classes. If $a \in A$ let ${}^a\chi_i$ denote its effect on χ_i, and similarly let $\mathcal{C}_i^a$ be its effect on $\mathcal{C}_i$. We assume

$$^a\chi_i(\mathcal{C}_j) = \chi_i(\mathcal{C}_j^a),$$

where $\chi_i(\mathcal{C}_j)$ means $\chi_i(x_j)$ for some representative $x_j \in \mathcal{C}_j$. (It is convenient to use a left action on characters and a right action on conjugacy classes, hence the notation $\mathcal{C}^a$.) Then the permutation characters of these two actions (on characters and on conjugacy classes) are the same.

Proof

To prove this, let X be the character table interpreted as a square matrix, that is, X is the matrix $(\chi_i(\mathcal{C}_j))$. Let θ_{char} and θ_{cc} be the permutation characters for the actions on characters and conjugacy classes.

If $a \in A$ then A affects a permutation of the characters, that is, the rows of X, and so it changes the character table to $P_a X$ where P_a is a permutation matrix. The permutation character is $\theta_{\mathrm{char}}(a) = \mathrm{tr}(P_a)$. On the other hand

$$^{a}\chi_i(\mathcal{C}_j) = \chi_i(\mathcal{C}_j^a),$$

where on the right hand side, we've permuted the columns of X, so qe obtain an identity

$$P_a X = X Q_a$$

where Q_a is also a permutation matrix. Now $\theta_{\mathrm{cc}}(a) = \mathrm{tr}(Q_a)$. The

Application of Brauer's Lemma to Frobenius groups

Proposition

Let G be a Frobenius group with kernel K and complement H. Let τ be a nontrivial character of K, and $1 \neq h \in H$. Define ${}^h\tau$ to be the character ${}^h\tau(k) = \tau(h^{-1}gh)$, which is the character of the representation ${}^h\pi(k) = \pi(h^{-1}gh)$. Then ${}^h\tau \neq \tau$.

By our last result, the action of h on the set of conjugacy classes of K by conjugation has only one fixed point, which is the identity. We may now apply Brauer's Lemma. Let θ_{char} and θ_{cc} be the permutation characters for the actions of H on characters and conjugacy classes. We've shown $\theta_{\text{cc}}(h) = 1$, since h fixes the identity conjugacy class and no other. So $\theta_{\text{char}}(h) = 0$, which implies that in its action on characters, also by conjugation, h fixes exactly one character, which must be the trivial character. In particular, ${}^h\tau \neq \tau$.

Main theorem

Theorem ((Restatement of the main theorem))

Let G be a Frobenius group and let τ be a nontrivial irreducible character of the Frobenius kernel K. Then τ^G is an irreducible representation of G. If τ' is another irreducible character of K then $\tau^G = (\tau')^G$ if and only if $\tau' = {}^h\tau$. The $|H|$ characters ${}^h\tau$ are all distinct, so induction gives an $|H|$-to-one map from irreducible nontrivial characters of K to characters of G.

Recall that we are proving that if τ is a nontrivial irreducible character of K, then τ^G is an irreducible character of H. We need to know the value of $\tau^G(k)$ with $k \in K$. This equals

$$\tau^G(k) = \sum_{Kh \in K\backslash G} \dot{\tau}(hkh^{-1})$$

where we are summing over the right cosets Kh of K.

Proof, continued

We may take the coset representatives to be the elements of H. We may also replace $\dot{\tau}$ by τ since $hkh^{-1} \in K$. Thus

$$\tau^G(k) = \sum_{h \in H} {}^h\tau(k).$$

The ${}^h\tau$ are distinct by our last result.

Proof

Now let τ' be another nontrivial character of K. By Frobenius reciprocity

$$\langle \tau^G, (\tau')^G \rangle_G = \langle \tau^G, \tau' \rangle_K = \sum_{h \in H} \langle {}^h\tau, \tau' \rangle_H = \begin{cases} 1 & \text{if } t' = {}^h\tau \text{ for some } h \in H \\ 0 & \text{otherwise.} \end{cases}$$

Taking $\tau' = \tau$ we see that $\langle \tau^G, (\tau')^G \rangle_G = 1$, so τ is irreducible. We've also proved that $\tau^G = (\tau')^G$ if and only if $\tau' = {}^h\tau$ for some $h \in H$.

We now have all the irreducible characters

We now have two different methods of producing irreducible characters of G: we may extend any irreducible character of H, or we may induce any nontrivial irreducible character of K. The following result completely determines the irreducible representations of a Frobenius group

Theorem

Every irreducible character of G is either the extension of an irreducible character of H or the induction of a nontrivial irreducible character of K.

Counting the conjugacy classes

Let h_H be the number of conjugacy classes of H, and let h_K be the number of conjugacy classes of K. There are two types of conjugacy classes of G. First, those classes whose elements have fixed points. Each such class, intersected with H is a conjugacy class of H, so there are h_H of these. Second, there are the conjugacy classes of elements that have no fixed point. These are all contained in K, and there are $h_K - 1$ of these (since we are excluding the identity). However we have shown that H acts without fixed points on these, so these become $(h_K - 1)/|H|$ conjugacy classes.

We've counted

$$h_H + \frac{h_K - 1}{|H|}$$

conjugacy classes for G. But we've produced exactly this many irreducible representations, so we are done.

The group of order 21, revisited

The group of order 21:

$$\langle a, b | a^7 = b^3 = 1, bab^{-1} = a^2 \rangle$$

is a Frobenius group. We worked out its character table in the in-class notes to Lecture 12.

	1	a	a^2	b	b^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ
χ_4	3	γ	δ	0	0
χ_5	3	δ	γ	0	0

We can use this to illustrate the ideas from today's lecture.