Frobenius Groups (I)

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Frobenius Groups

The definition of a Frobenius Group in Dummit and Foote (page 896) is a nonstandard one. It can be shown that it is equivalent to the definition we will use. However to prove this equivalence is not easy.

Dummit and Foote define a Frobenius group to be a group G having a normal subgroup N such that for every $x \in N$ the centralizer $C(x) \subseteq N$.

Most references define a Frobenius group to be a group with a transitive group action on a set X such that no element of G except the identity fixes two elements. Proving that these definitions are the same is nontrivial.

Frobenius Groups

Definition

A Frobenius Group is a group G with a faithful transitive action on a set X such that no element fixes more than one point.

An action of G on a set X gives a homomorphism from G to the group of bijections X (the symmetric group $S_{|X|}$). In this definition faithful means this homomorphism is injective. Let H be the stabilizer of a point $x_0 \in X$. The group H is called the Frobenius complement.

Next week we will prove:

Theorem (Frobenius (1901))

A Frobenius group G is a semidirect product. That is, there exists a normal subgroup K such that $G = HK$ and $H \cap K = \{1\}$.

The mystery of Frobenius' Theorem

Since Frobenius' theorem doesn't require group representation theory in its formulation, it is remarkable that no proof has ever been found that doesn't use representation theory!

Web links:

- Frobenius groups (Wikipedia)
- Fourier Analytic Proof of Frobenius' Theorem (Terence Tao)
- Math Overflow page on Frobenius' theorem

There are many results in group theory that don't involve representations in their statements but whose proofs do involve representation theory. Frobenius' theorem is a very striking example.

Examples of Frobenius groups

Let G be a Frobenius group. Let H be the isotropy subgroup of a point in X. Then H is called a Frobenius complement. The normal subgroup K that is predicted by the theorem is called the Frobenius kernel

- Let F be a finite field. The group of affine transformations $x \to ax + b$ of F is a Frobenius group. The Frobenius kernel is the group of translations, isomorphic to the additive group of F.
- A_4 is a Frobenius group.
- Any dihedral group D_{2n} with n odd is a Frobenius group.

The Frobenius Kernel

Let G be a Frobenius group acting on a set X. Let H be the isotropy subgroup of a fixed point $x_0 \in X$.

Let K^* be the set of elements of G that do not fix any element of X. Furthermore, let $K = K^* \cup \{1\}$. The set K is called the Frobenius kernel.

A more precise statement of Frobenius' theorem is:

Theorem

The Frobenius Kernel K is a subgroup of G.

For example, if $G = A_4$, then $K = V$ is the four-group.

The cardinality of the Frobenius Kernel

Let us compute the cardinality of K.

If $x \in X$ let G_x be the isotropy subgroup, so $G_{x_0} = H$. Then

$$K = G - \bigcup_{x \in X} (G_x - \{1\}).$$

The sets $(G_x - \{1\})$ are disjoint, since by definition of a Frobenius group, only the identity element fixes more than one point. Each such set has cardinality $|H| - 1$. So

$$|K| = |G| - |X| \cdot (|H| - 1) = |G| - |X| \cdot |H| + |X|.$$

Now (by the orbit stabilizer theorem) $|X| \cdot |H| = |G|$, so $|K| = |X|$.

The Frobenius group is a semidirect product

Suppose we know Frobenius's theorem, that K is a subgroup of G. It is obviously normal, and $K \cap H = \{1\}$. Since $|K| = |X| = [G : H]$, it follows that G is a semidirect product.

The mystery of Frobenius' theorem is that there is no obvious reason for K to be a subgroup of G!

The only known proofs of this fact use character theory.

The Frobenius group of order 20

Let us compute the character table of a Frobenius group. We will consider the Frobenius group of order 20, which is isomorphic to the normalizer of a 5-Sylow subgroup in S_5. We can realize it as the group of matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{F}_5, a \neq 0 \right\}.$$

It has generators

$$h = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \qquad k = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

that satisfy $h^4 = 1$, $k^5 = 1$, $hkh^{-1} = k^2$. Let $H = \langle h \rangle$, $K = \langle k \rangle$. This group is a Frobenius group with Frobenius complement H and Frobenius kernel K.

Conjugacy classes, derived group

Thus

$$G = \langle h, k | h^4 = k^5 = 1, hkh^{-1} = k^2 \rangle.$$

We find the following conjugacy classes

size	1	4	5	5	5
representative	1	k	h	h^2	h^3

Since G/K is abelian, the commutator subgroup G' is contained in K. It cannot be any smaller than K since this would make it trivial and G would be abelian. Therefore $G' = K$. The quotient G/K is cyclic of order 4, generated by the coset hK, and we obtain four linear characters by pulling back the characters of $G/K \cong Z_4$.

Linear characters

There are 5 conjugacy classes and hence 5 irreducible representations. Since $\sum d_i^2 = 20$ the remaining irreducible χ_5 has degree 4. Therefore we have this much of the character table:

	1	4	5	5	5
	1	k	h	h^2	h^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4				

We could get the last character by decomposing the regular representation of G.

The nonlinear character

The last remaining character χ_5 has degree 4. We could obtain its value by decomposing the regular representation, or from the following considerations.

Since χ_2 is linear, $\chi_2\chi_5$ is another irreducible representation of degree 4, and since there is only one, $\chi_2\chi_5 = \chi_5$. This tells us that χ_5 vanishes where ever χ_2 is nonzero, so the only nonzero values of χ_5 are at 1 and k. We know $\chi_5(1) = 4$ and since k has 4 conjugates, the orthogonality relation $\langle \chi_5, \chi_1 \rangle = 0$ implies that $\chi_5(k) = -1$.

The complete character table

	1	4	5	5	5
	1	k	h	h^2	h^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

The nonlinear character is induced from K

Let us show that χ_5 is an induced character. Because its degree is 4, it might be induced from a linear representation of the unique subgroup K of index 4. We could take any nontrivial linear character of K, so let us take the character $\psi(k^m) = \zeta^m$ where $\zeta = e^{2\pi i/5}$. To calculate ψ^G, we need coset representatives for $N\backslash G$. We may take these to be the elements of H, and

$$\psi^G(g) = \dot\psi(g) + \dot\psi(hgh^{-1}) + \dot\psi(h^2gh^{-2}) + \dot\psi(h^3gh^{-3}).$$

If $g \notin K$ then since K is normal, no conjugate of g can be in K and $\psi^G(g) = 0$. On the other hand, if $g = 1$ this formula gives the right value 4, and if $g = k$ it gives

$$\zeta + \zeta^2 + \zeta^4 + \zeta^3 = -1.$$

We see that $\chi_5 = \psi^G$.

An observation

Here is the character table for the group $H \cong Z_4$.

	1	h	h^2	h^3
χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_3	1	-1	1	-1
χ_4	1	$-i$	-1	i

Comparing this with the character table for G we see that every irreducible character of H (and hence, indeed, any character) can be extended to a character of G.

Of course there is a simple reason for this since $G = H \ltimes K$ we have an isomorphism $G/K \cong H$, hence the character can be pulled back under the projection map

$$G \longrightarrow G/K \cong H.$$

Arguing backwards

This argument requires knowledge that K is a normal subgroup. Suppose that we are attempting to prove Frobenius's theorem. We have a group action on a set X and an isotropy subgroup H. Although we have a definition of K as a set, we do not know that it is a group. So we cannot argue as above.

But using character theory, we will prove (next week) that every representation of H can be extended to a representation of G.

Then we may start with any faithful representation π of H, extend it to G, and check that the kernel of π is

$$K = \{1\} \cup \{k \in G | k \text{ has no fixed points}\} .$$

It will follow that $K = \ker(\pi)$ is a normal subgroup. This is the strategy to prove Frobenius' theorem.

The Dummit and Foote definition of a Frobenius group

Dummit and Foote give the following definition of a Frobenius group. They define a Frobenius group G to be a group with a proper, nontrivial normal subgroup K such that if $x \in K$ and $x \neq 1$ then the centralizer $C(x) \subseteq K$.

Theorem

Let G be a group with a proper, nontrivial normal subgroup K such that if $x \in K$ and $x \neq 1$ then the centralizer $C(x) \subseteq K$. Then there exists a set X with cardinality $|K|$ and an action of G on X such that G is a Frobenius group with Frobenius kernel K.

We must produce the Frobenius complement H or equivalently, the set X with its group action.

K is a normal Hall subgroup

Proposition

Let G be a group with a proper, nontrivial normal subgroup K such that if $x \in K$ and $x \neq 1$ then the centralizer $C(x) \subseteq K$. Then $|K|$ and $[G : K]$ are coprime.

If not, there is a prime p that divides both $|K|$ and $[G : K]$. Let P_K be a p-Sylow subgroup of K. Find a Sylow subgroup P of G containing P_K. Thus P contains non-identity elements in K and others not in K. As a p-group it has a nontrivial center (Theorem 1 in Dummit and Foote, page 188). Let $x \neq 1$ be a central element. There are two cases. If $x \in K$, then since $P \subset C(x)$ contains elements that are not in K, the assumptions about K are violated. On the other hand, if $x \notin K$, let y be a nontrivial element of P that is in K. Then $x \in C(y)$ but $x \notin K$, so again, the assumptions are violated.

The Schur-Zassenhaus theorem

Now we need to make use of a well-known theorem that we will not prove.

Theorem (Schur-Zassenhaus)

Let G be a group and K a normal subgroup such that $|K|$ and $[G : K]$ are coprime. Then G contains a complementary subgroup H such that $H \cap K = 1$ and $HK = G$. Thus $G = H \ltimes K$.

This famous result is Theorem 39 in Chapter 17 of Dummit and Foote, page 829. The proof uses some basic group cohomology. This theorem appeared in a book by Hans Zassenhaus, who attributed it to Schur. It is usually called the Schur-Zassenhaus theorem.

Returning to the proof of the theorem

Returning to the group such that $1 \neq x \in K$ implies $C(x) \subseteq K$, the Schur-Zassenhaus theorem implies that there is a subgroup H such that $G = HK$ and $H \cap K = 1$. Now let X be the space G/H of left cosets xH.

To show that G is a Frobenius group we must show that if $g \in G$ has two fixed points in X then $g = 1$.

G is a Frobenius group

We note that every coset gH has a representative in K since $KH = G$. Thus there are two distinct elements k_1 and k_2 of K such that

$$gk_1H = k_1H, \qquad gk_2H = k_2H.$$

Let $h = k_1^{-1}gk_1$. Then $hH = H$ so $h \in H$. Let $k = k_1^{-1}k_2$. Thus $1 \neq k \in K$ and

$$k_1hk_1^{-1}k_2H = gk_2H = k_2H, \qquad k^{-1}hkH = H$$

so $k^{-1}hk \in H$.

The end of the proof

Now consider the commutator

$$h^{-1}k^{-1}hk.$$

Since we've proved that $k^{-1}hk \in H$ we have $h^{-1}k^{-1}hk = h^{-1}(k^{-1}hk) \in H$. On the other hand since K is normal

$$h^{-1}k^{-1}hk = (h^{-1}kh)^{-1}k \in K.$$

The commutator is therfore in $H \cap K$ and so $h^{-1}k^{-1}hk = 1$. This shows that h and k commute. Since $C(k) \subseteq K$ by hypothesis we have $h \in H \cap K$ and thus $h = 1$. Thus $g = k_1hk_1^{-1} = 1$. We have proved that G is a Frobenius group.