

# Stanford PhD algebra qualifying exam material

July 13, 2019

The following sources are references for the assumed background listed below, as well as everything in 210A and (essentially all of) 210B:

[J1], [J2] N. Jacobson, *Basic Algebra I, II*, 2nd ed., Dover, 1985.

[DF] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed, Wiley, 2004.

[L] S. Lang, *Algebra* (3rd ed.), Graduate Texts in Math vol. 211, Springer, 2002.

The courses 210A and 210B cover the material below that is not within the “assumed background” list. These courses may also cover additional material (chosen by the specific instructor, so varying from year to year) to fulfill the additional educational purpose of being graduate-level introductory courses in algebra.

## Assumed background.

The material listed below within the following 3 core topics constitute a mixture of concepts that *we expect you have learned during your undergraduate studies or have acquired on your own from the indicated (or other) references before arriving at Stanford*. Make sure to understand examples and work out exercises (provided in abundance in [DF] and [J1] for group theory and in [L] and [J1] for field theory).

The experience of earlier cohorts of graduate students indicates that it is much better to **acquire the assumed algebra background prior to your arrival at Stanford** rather than planning to squeeze it in amidst other things during the academic year or during the breaks after you are here (as the latter is **too little time** to absorb many new concepts in practice). In particular, if you aim to pass the analysis qual upon arriving then we recommend to combine that preparation with filling any gaps in your algebra background **before you enroll**.

## GROUP THEORY

Normal subgroups, simple groups, and Jordan–Holder theorem ([J1, 1.8, 4.6], [L, I.3]), group actions on sets (including symmetric and alternating groups, and simplicity of  $A_n$  ( $n \geq 5$ )) ([J1, 1.6-1.7, 1.12, 4.6], [L, I.5]), Sylow theorems and applications (including solvability of  $p$ -groups) ([J1, 1.13, 4.6], [L, I.6]), semi-direct products [DF, §5.5], free groups and group presentations ([J1, 1.11], [DF, §6.3]).

## LINEAR ALGEBRA

Characteristic polynomials, eigenvalues and eigenvectors, dual space, trace and determinant, diagonalization. ([DF, Ch. 11], [L, Ch. XIII])

## FIELD THEORY

Degree of a field extension, splitting field, finite fields, theorem of the primitive element, field embeddings and automorphisms, Galois theory with low-degree worked examples. ([J1, Ch. 4], [DF, Ch. 13–14], [L, Ch. V–VI])

## 210A Course material

## BASIC COMMUTATIVE RINGS

Principal ideal domains, unique factorization domains, polynomial and power series rings, prime and maximal ideals, Chinese Remainder Theorem for comaximal ideals. Local rings and localization, Nakayama’s lemma, Noetherian rings, Hilbert

basis theorem. ([J1, 2.10-2.12, 2.14-2.16], [J2, 7.1-7.4, 7.8, 7.10], [DF, Ch. 7-9, 15], [L, Ch. II, IV, X.1, X.4])

#### MODULES AND HOMOLOGICAL METHODS

Finitely generated modules over PID's, applications (Jordan and rational canonical form); chain conditions (esp. Noetherian modules), tensor product and Hom, flatness, categories and functors (examples, universal properties, representability, adjointness), projective/injective resolutions, Tor and Ext. ([J1, 3.6-3.10], [J2, 1.1-1.6, 3.2, 3.7, 3.10-3.11, 6.1-6.8], [DF, Ch. 10, 12, 17], [L, Ch. III, X, XVI, XX])

#### ADVANCED LINEAR ALGEBRA

Minimal polynomial, Jordan decomposition for endomorphisms and automorphisms, multilinear algebra, tensor algebra (tensor powers, symmetric and exterior powers). ([J1, 6.1-6.4], [DF, §11.5, Ch. 12], [L, Ch. XIII-XIV])

#### ADDITIONAL LINEAR ALGEBRA (FOR SELF-STUDY IF NOT COVERED)

Bilinear forms (symmetric, alternating) over fields, dictionary between quadratic and symmetric bilinear forms away from char. 2, signature of quadratic forms over  $\mathbf{R}$  (Sylvester's Law), Hermitian forms over  $\mathbf{C}$ , spectral theorem for Hermitian forms over  $\mathbf{C}$  and symmetric bilinear forms over  $\mathbf{R}$ , structure of symplectic forms over fields. ([L, Ch. XV.1, XV.3-XV.8], [J1, 6.3])

### 210B Course material

#### ADVANCED FIELD THEORY (approx. 2 weeks)

Algebraic extensions and algebraic closure, separable and inseparable extensions (not on qual), norm and trace, transcendence degree, finitely generated extensions of a field. ([J2, 8.1, 8.7, 8.12, 8.15-8.16], [DF, §14.9], [L, Ch. V-VI, VIII.1])

#### COMMUTATIVE ALGEBRA AND AFFINE ALGEBRAIC GEOMETRY (approx. 5 weeks)

Nullstellensatz, affine algebraic sets and irreducible components (over an algebraically closed field),  $\text{Spec}(A)$  and Zariski topology. Integrality, integral closure and finiteness properties, Noether normalization (over infinite fields), going-up and going-down (for integral extensions). ([J2, 7.5-7.7, 7.12, 8.13], [DF, Ch. 15], [L, Ch. VII, VIII.2, IX])

Dimension theory for finitely generated algebras over a field: relation to transcendence degree, properties, examples. ([J2, 8.13], Chapter 5 in *Commutative Ring Theory* by Matsumura)

Artinian rings, Dedekind domains, discrete valuation rings (with examples). ([J2, 7.11, 10.1-10.3], [DF, Ch. 16], Ch. III in *Algebraic Theory of Numbers* by Samuel)

#### REPRESENTATION THEORY (FINITE GROUPS, CHAR. 0 FOR QUAL) (approx. 3 weeks)

Notions of group algebra and group representation over a field, irreducible representations of finite groups, Schur's lemma, Maschke's theorem. Interaction of representations with tensorial operations, characters and orthogonality (only in characteristic 0), examples over  $\mathbf{C}$  with small groups. ([J2, 5.1-5.2, 5.4-5.6], [DF, 18.1], [L, Ch. XVIII], Ch. 1-2 in *Linear Representations of Finite Groups* by Serre, Ch. 1-2 in *Representation Theory: a first course* by Fulton and Harris)

Character tables, induced representations, Frobenius reciprocity (examples over  $\mathbf{C}$ ). ([J2, 5.5, 5.9-5.11], Chapters 3, 5, 7 in Serre, Chapter 3 in Fulton-Harris)