The Zariski Topology

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Much of the motivation for commutative algebra comes from algebraic geometry. Hence the study of commutative algebra is best done with a few ideas of algebraic geometry in hand. Affine space, or more generally an affine variety, has a topology, called the *Zariski topology*. It works best if the ground field is algebraically closed (so we can use the Nullstellensatz). Reference: Lang's *Algebra*, Section IX.2.

1 Zariski topology of affine space

Let F be a field, and let $\mathbb{A}^n(F)$ be affine n-space. This is F^n as a set. The *Zariski topology* is a topology on $\mathbb{A}^n(F)$. Let $R = F[x_1, \cdots, x_n]$ be the polynomial ring in n variables. If $f \in R$ we regard f as a function on $\mathbb{A}^n(F)$. If $S \subseteq R$ is any subset, let

$$V(S) = \{a \in \mathbb{A}^n(F) | f(a) = 0 \text{ for all } f \in S\}.$$

Let $\mathfrak{a} = \langle S \rangle$ be the ideal generated by S. Then clearly

$$V(S) = V(\mathfrak{a}).$$

Moreover since $f^n(a) = 0$ if and only if $f(a) = 0$ we see that

$$V(\mathfrak{a}) = V(r(\mathfrak{a})).$$

Let us call a subset $X \subseteq \mathbb{A}^n(F)$ *closed* or *Zariski-closed* if $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. A closed subset of $\mathbb{A}^n(F)$ is also called an *algebraic set*. (This is the terminology used in Lang, Section IX.2.)

Proposition 1 *If X_i $(i \in I)$ is a family of closed sets then $\bigcap_{i \in I} X_i$ is closed. If X and Y are closed, then $X \cup Y$ is closed. Therefore the Zariski closed sets form the set of closed sets in a topology.*

Proof This will be a homework problem in the second problem set. □

2 The radical of an ideal

Let $\mathfrak{a}$ be an ideal in a commutative ring R. Let $r(\mathfrak{a})$ be the *radical* of $\mathfrak{a}$ defined by

$$r(\mathfrak{a}) = \{f \in R | f^n \in \mathfrak{a} \text{ for sufficiently large } n\}.$$

Proposition 2 *The radical $r(\mathfrak{a})$ is an ideal. Moreover $r(r(\mathfrak{a})) = r(\mathfrak{a})$.*

Proof To show that $r(\mathfrak{a})$ is an ideal, we need to check that if $f_1, f_2 \in r(\mathfrak{a})$ then $f_1+f_2 \in r(\mathfrak{a})$. For this, there exist integers n_1 and n_2 such that $f_1^{n_1}, f_2^{n_2} \in \mathfrak{a}$. By the binomial theorem $(f_1+f_2)^{n_1+n_2}$ is a linear combination of terms $f_1^a f_2^b$ where $a+b = n_1+n_2$. So either $a \geqslant n_1$ and $f_1^a \in \mathfrak{a}$ or $b \geqslant n_2$ and $f_2^b \in \mathfrak{a}$. Either way, $f_1^a f_2^b \in \mathfrak{a}$ so $f_1 + f_2 \in r(\mathfrak{a})$. It is obvious that $Rr(\mathfrak{a}) \subseteq r(\mathfrak{a})$ and that $0 \in r(\mathfrak{a})$ and so $r(\mathfrak{a})$ is an ideal. It is obvious from the definition that $r(r(\mathfrak{a})) = r(\mathfrak{a})$. □

An ideal $\mathfrak{a}$ is *radical* if $\mathfrak{a} = r(\mathfrak{a})$. For example, if X is a closed subset of $\mathbb{A}^n(F)$ and $\mathfrak{a}$ is the ideal of $f \in F[x_1, \cdots, x_n]$ then it is obvious that $\mathfrak{a}$ is a radical ideal. As a special case, $\mathfrak{N} := r((0))$ is the set of nilpotent elements of R. It is called the *nilradical* of R.

Proposition 3 *The nilradical $\mathfrak{N}$ is the intersection of all prime ideals in R.*

Proof See Lang, X.2.2 on page 417. □

A commutative ring A is called *reduced* if its nilradical is zero. If $\mathfrak{a}$ is an ideal in the commutative ring R, then it is easy to see that $\mathfrak{a}$ is radical if and only if $R/\mathfrak{a}$ is reduced.

3 The Nullstellensatz

Example 1 Suppose $F = \mathbb{R}$ and $S = \{x_1^2 + x_2^2 + 1\} \subseteq \mathbb{R}[x_1, x_2]$. Then $V(S) = \varnothing$.

This cannot happen if F is an algebraically closed field.

Theorem 1 (Weak Nullstellensatz) *Suppose that F is algebraically closed. Let $\mathfrak{a}$ be an ideal of $R = F[x_1, \cdots, x_n]$. If $V(\mathfrak{a}) = \varnothing$ then $\mathfrak{a} = R$.*

Theorem 2 (Strong Nullstellensatz) *Suppose that F is algebraically closed. Let $\mathfrak{a} \subseteq F[x_1, \cdots, x_n]$. Then the ideal of $f \in F[x_1, \cdots, x_n]$ that vanish on $V(\mathfrak{a})$ is $r(\mathfrak{a})$.*

These results will be proved in week 3.

4 Affine varieties

Assume that F is algebraically closed. Let X be a Zariski closed subset of $\mathbb{A}^n(F)$. Then X inherits a topology from $\mathbb{A}^n(F)$, just as any subset of a topological space inherits a topology. This can also be understood as a Zariski topology.

Indeed, let $\mathfrak{a}$ be the ideal of $f \in F[x_1, \cdots, x_n]$ that vanish on X, and let $A = F[x_1, \cdots, x_n]/\mathfrak{a}$. We may regard elements of A as the ring of polynomial functions on X. The ring A is reduced, Noetherian and finitely generated as an A-algebra. If $\mathfrak{a}$ is prime, then A is an integral domain. If this is true, then we call X an *affine algebraic variety*. Now the correspondence between ideals and algebraic subsets of X goes through just as for $\mathbb{A}^n(F)$: if $S \subset A$ we may consider $V(S) = \{x \in X | f(x) = 0 \text{ for } f \in S\}$. A subset Y of X is closed if and only if it is $V(S)$ for some S. This is equivalent to Y being a closed subset of $\mathbb{A}^n(F)$ that is contained in X.