Affine Varieties

1 Affine algebraic sets and their Zariski topology

There is a close association between affine varieties and certain rings. An ideal $\mathfrak{a}$ of a ring is called *radical* if $\mathfrak{a} = r(\mathfrak{a})$. A commutative ring A is called *reduced* if it has no nilpotent elements. It is clear that if $\mathfrak{a}$ is a radical ideal then $A/\mathfrak{a}$ is reduced.

We continue from the previous article *Zariski topology*. Let F be an algebraically closed field (so we can use the Nullstellensatz). Let $X \subseteq \mathbb{A}^n(F)$ be a Zariski closed set. Thus $X = V(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal. Since $V(\mathfrak{a}) = V(r(\mathfrak{a}))$ we may assume that $\mathfrak{a}$ is a radical ideal.

A topological space X is called *reducible* if we may write $X = Y \cup Z$ where Y and Z are proper closed subsets. For example, let $X = V(S) \subseteq \mathbb{A}^2(F)$ where $S = \{x_1 x_2\}$. Then $X = Y \cup Z$ where Y and Z are the coordinate axes, so this space is reducible.

Proposition 1. *Let $\mathfrak{a}$ be a radical ideal. The Zariski closed subset $V(\mathfrak{a})$ is irreducible if and only if $\mathfrak{a}$ is prime.*

Proof. This is Theorem IX.2.3 on page 382 of Lang's *Algebra*. It is also homework problem (to be assigned in Week 3), but if you look at the proof in Lang, when you do the homework problem note that he doesn't explain why $A \neq V$ and $B \neq V$. Either write your own solution without looking at the proof in Lang, or do look at the proof in Lang but make sure you discuss this point clearly. □

Let $R = F[x_1, \cdots, x_n]$ be the polynomial ring, regarded as functions on $\mathbb{A}^n(F)$. A function on the closed set $X \subseteq \mathbb{A}^n(F)$ is called *polynomial* if it is the restriction of a polynomial in R. Let $\mathcal{O}(X)$ be the ring of polynomial functions on X. Then clearly $\mathcal{O}(X) \cong R/\ker(\varphi)$ where $\varphi : R \longrightarrow \mathcal{O}(X)$ is the restriction homomorphism. The kernel $I(X) := \ker(\varphi)$ is the ideal of $f \in F[x_1, \cdots, x_n]$ that vanish on X, and by the Nullstellensatz (since F is algebraically closed and $\mathfrak{a} = r(\mathfrak{a})$) we have $\ker(\varphi) = \mathfrak{a}$. Thus $\mathcal{O}(X) \cong R/\mathfrak{a}$. This ring is called the *coordinate ring* or *affine algebra* of X.

Definition 2. *An affine algebra over F is a commutative algebra that is finitely generated and reduced.*

Note that an affine algebra is Noetherian, since it is a quotient of a polynomial algebra over F, which is Noetherian by the Hilbert basis theorem.

Proposition 3. *Let A be an affine algebra over the algebraically closed field F. Then $A = \mathcal{O}(X)$ for some affine algebraic set X. If A is an integral domain, then X is a variety.*

Proof. Let $a_1, \cdots, a_n$ be generators of A as an algebra over F. Then there is a homomorphism from the polynomial ring $F[x_1, \cdots, x_n]$ to A mapping x_i to a_i. Let $\mathfrak{a}$ be the kernel of this homomorphism, so $A \cong F[x_1, \cdots, x_n]/\mathfrak{a}$. Since A is reduced, it is easy to see that $\mathfrak{a}$ is a radical ideal and if $X \cong V(\mathfrak{a})$ then $A \cong \mathcal{O}(X)$. □

We will call a Zariski closed set X a *variety* if it is irreducible, or equivalently if $\mathcal{O}(X)$ is an integral domain. If it is not necessarily irreducible, we will call the closed set X an *affine algebraic set* (or, equivalently, we can just refer to it as a Zariski closed set). Terminologies differ: some authors do not require varieties to be irreducible.

If $X \subset \mathbb{A}_n(F)$ and $Y \subset \mathbb{A}_m(F)$ are affine algebraic sets, by a *morphism* $f : X \to Y$ we mean a polynomial map. Thus we require m polynomials $f_i \in F[X_1, \cdots, X_n]$ such that for $a = (a_1, \cdots, a_n) \in \mathbb{A}^n(F)$ we have

$$f(a_1, \cdots, a_m) = (f_1(a_1, \cdots, a_n), \cdots f_m(a_1, \cdots, a_n)).$$

It is required that this map takes X into Y. The polynomials f_i may not be uniquely determined since the map is unchanged if we change f_i by an element of the ideal $I(X)$. With this notion of morphism, affine algebraic sets form a category.

Now that we consider affine algebraic sets a category, it makes sense to loosen the connection with affine spaces. We started by defining an algebraic set as a subset $X = V(\mathfrak{a})$ of $\mathbb{A}^n$, which itself has a topology (the Zariski topology). Of course X inherits a topology from $\mathbb{A}^n$, also called the Zariski topology. But if we chose a different embedding of X into some other affine space $\mathbb{A}^m$, it would also inherit another Zariski topology from $\mathbb{A}^m$, and it turns out these two topologies are the same. An easy way to see that is to note that the correspondence between closed subsets of $\mathbb{A}^n$ and radical ideals of $\mathcal{O}(\mathbb{A}^n) = F[x_1, \cdots x_n]$ can be extended directly to a correspondence between closed subsets of X (that is, closed subsets of $\mathbb{A}^n$ that are contained in X) and radical ideals of $\mathcal{O}(X) = F[x_1, \cdots, x_n]/\mathfrak{a}$. Indeed, a closed subset $Z = V(\mathfrak{b})$ of $\mathbb{A}^n$ is contained in X if and only if $\mathfrak{a} \subseteq \mathfrak{b}$ and so $\mathfrak{c} = \mathfrak{b}/\mathfrak{a}$ is an ideal of $\mathcal{O}(X)$.

Therefore we may define the Zariski topology on X directly without reference to its embedding in a particular affine space. If $\mathfrak{c}$ is a radical ideal of $\mathcal{O}(X)$, we may define $V(\mathfrak{c}) \subseteq X$ to be the set of $a \in X$ such that $f(a) = 0$ for all $f \in \mathfrak{c}$, and these will be the closed sets in X. Once we see that the Zariski topology in X can be defined this way, we see that it is independent of the embedding of X into an affine space $\mathbb{A}^n$.

2 Functorial properties of the affine algebra

Let $X \subseteq \mathbb{A}^n(F)$ and $Y \subseteq \mathbb{A}^m(F)$ be affine algebraic sets. A *polynomial map* $f : X \longrightarrow Y$ is any map induced from a polynomial mapping from $\mathbb{A}^n(F) \longrightarrow \mathbb{A}^m(F)$.

Composition with f gives an algebra homomorphism $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, so the affine algebra is a contravariant functor, from the category of affine algebraic sets to the category of affine algebras, or from the category of affine varieties to the category of affine algebras that are integral domains.

In the following examples, the affine algebraic sets are all varieties, so their affine algebras are integral domains.

Example 4. *Let $X = \mathbb{A}^1$ and $Y = \mathbb{A}^2$, and define $f : X \longrightarrow Y$ by $f(a) = (a, 0)$. The map f is injective but not surjective. The affine algebras are polynomial rings, $\mathcal{O}(X) = F[x]$ and $\mathcal{O}(Y) = F[x, y]$, and $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is the map $x \to x$, $y \to 0$. This map is surjective but not injective.*

Example 5. *Let $X = \mathbb{A}^2$ and $Y = \mathbb{A}^1$ and define $f : X \longrightarrow Y$ by $f(a, b) = a$. The map f is surjective but not injective. The map on affine algebras $F[x] \longrightarrow F[x, y]$ is the inclusion which is injective but not surjective.*

One might think from these examples that f is injective if and only if f^* is surjective and vice versa, but the next example shows that this is not the case.

Example 6. *Let X be the hyperbola $\{(a, b)|ab = 1\} \subseteq \mathbb{A}^2$ and let $Y = \mathbb{A}^1$. Define a map $f : X \longrightarrow Y$ by $f(a, b) = a$. This map is injective but not surjective. Now $\mathcal{O}(X) = F[x_1, x_2]/(x_1x_2 - 1)$ where $F[x_1, x_2] = \mathcal{O}(\mathbb{A}^2)$ is the polynomial ring; if x, y are the images of x_1 and x_2, then $y = x^{-1}$ so $\mathcal{O}(X) = F[x, x^{-1}]$. On the other hand, $\mathcal{O}(Y) = F[x]$ and $f^* : F[x] \longrightarrow F[x, x^{-1}]$ is the inclusion map, which is injective but not surjective.*

This example shows that the naive expectation that f is injective if and only if f^* is surjective and vice versa is incorrect.

There is an easy criterion for f^* to be injective. Let us say that $f : X \longrightarrow Y$ is *dominant* if $f(X)$ is dense in Y in the Zariski topology. The map in Example 6 is dominant but not surjective.

Proposition 7. *Let X and Y be varieties and $f : X \longrightarrow Y$ a morphism. Then $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is injective if and only if f is dominant.*

Proof. if $\phi \in \mathcal{O}(Y)$ define $Y_\phi = \{b \in Y|\phi(b) \neq 0\}$. This is an open set since its complement is closed, and Y_ϕ is called a *principal open set*. It is easy to see that these are a basis of the topology, meaning that every open set is a union of principal open sets. So $f(X)$ is dense if and only if it meets every principal open set. Now $f(X) \cap Y_\phi = \varnothing$ if and only if ϕ vanishes on $f(X)$, that is, if $f^*(\phi) = 0$. So $f(X)$ fails to be dense if and only if f^* has a nontrivial kernel. □

We assume that X and Y are varieties and $f : X \longrightarrow Y$ is a dominant morphism, so that $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is injective. We may identify $\mathcal{O}(Y)$ as a subring of $\mathcal{O}(X)$. The next Proposition explains the geometric meaning of a prime being above another prime, in the case of maximal ideals.

Proposition 8. *Let $f : X \longrightarrow Y$ be a dominant map. Let $A = \mathcal{O}(Y)$ and $B = \mathcal{O}(X)$. Identify A as a subring of B via the injective ring homomorphism f^*. Let $y \in Y$ and let $\mathfrak{p}$ be the maximal ideal of A consisting of functions that vanish at y. Similarly let $\mathfrak{P}$ be the prime of B consisting of functions that vanish at x. Then $f(x) = y$ if and only if $\mathfrak{P} \cap A = \mathfrak{p}$.*

Proof. Suppose that $f(x) = y$ and let $\phi \in \mathfrak{p}$. The image of ϕ in B is $f^*(\varpi) = \phi \circ f$, which then vanishes at x. We are identifying ϕ with its image under f^*, so with this identification $\phi \in B$. This shows that $\mathfrak{p} \subseteq A \cap \mathfrak{P}$, but since $\mathfrak{p}$ is maximal, $\mathfrak{P} \cap A = \mathfrak{p}$. We leave the other direction to the reader. □

Proposition 9. *Let $f : X \longrightarrow Y$ be a dominant morphism, and assume that $\mathcal{O}(X)$ is integral over $\mathcal{O}(Y)$. Then f is surjective.*

Proof. We will use the notations $A = \mathcal{O}(Y)$, $B = \mathcal{O}(X)$ and identify A as a subring of B via the injection $f^* : A \longrightarrow B$. Let $y \in Y$ and let $\mathfrak{p}$ be the maximal ideal of A consisting of functions vanishing at y. Since B is integral over A, there exists a maximal ideal $\mathfrak{P}$ of B above $\mathfrak{p}$. (Lang, Propositions 1.10 and 1.11 in Chapter VII, page 339.) Now by the Nullstellensatz, $\mathfrak{P}$ consists of all functions that vanish at some point $x \in X$. Then $f(x) = y$ by Proposition 8. □

In view of this, the failure of the dominant map f in Example 6 is related to the fact that $F[x, x^{-1}]$ is not integral over $F[x]$.

3 More on irreducibility

Earlier in these notes we defined a topological space X to be irreducible if it is not a union of two proper closed subsets.

Proposition 10. *An affine algebraic set X is irreducible if and only if $\mathcal{O}(X)$ is an integral domain.*

Proof. Embed X into affine space $\mathbb{A}^n$, so $\mathcal{O}(X) = F[X_1, \cdots, X_n]/\mathfrak{a}$ for some radical ideal $\mathfrak{a}$. By Proposition 1, X is irreducible if and only if $\mathfrak{a}$ is prime, that is, if and only if the quotient $F[X_1, \cdots, X_n]/\mathfrak{a}$ is an integral domain. □

A topological space is called *Noetherian* it satisfies the descending chain condition for closed subsets. Thus every descending chain of closed sets $X_1 \supseteq X_2 \supseteq \cdots$ must eventually terminate: $X_n = X_{n+1} = \cdots$ for sufficiently large n.

Proposition 11. *If X is a Noetherian space, then any set of closed subsets of X has a minimal element.*

Proof. This follows immediately from Zorn's Lemma. Let Σ be the set of closed subsets of X, ordered by inclusion. By the definition of a Noetherian space, every totally ordered subset has a lower bound, so by Zorn's Lemma, Σ has minimal elements. □

Proposition 12. *An affine algebraic set is a Noetherian topological space with the Zariski topology.*

Proof. Let X be an affine algebraic set. The Zariski closed subsets of X are in bijection (inclusion reversing) with the radical ideals of $\mathcal{O}(X)$, which is a Noetherian ring. Since $\mathcal{O}(X)$ thus satisfies the ascending chain condition for ideals, X satisfies the descending chain condition for closed subsets. □

Here we prove a basic property for Noetherian spaces.

Proposition 13. *Let X be a Noetherian topological space. Then X admits a finite decomposition into closed subsets:*

$$X = X_1 \cup \cdots \cup X_n.$$

We may clearly assume that there are no inclusion relations among the X_i, so if $X_i \subseteq X_j$ then $i = j$. With this assumption, the decomposition is unique.

Proof. It is clear that a closed subset of a Noetherian space is irreducible. Let Σ be the set of all closed subspaces of X that do *not* have such irreducible decompositions. We will show that Σ is empty. If not, then Σ has a minimal element Y by Proposition 11. Clearly Y cannot be irreducible, so write $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are proper closed subspaces of Y. By the minimality of Y, it must be true that Y_1 and Y_2 is each the finite union of irreducible subspaces, but then so is $Y = Y_1 \cup Y_2$, which is a contradiction.

For the uniqueness relation, let $X = Y_1 \cup \cdots \cup Y_m$ be another such decomposition. We claim that every Y_i is contained in some X_j. Indeed,

$$Y_i = \bigcup_j (Y_i \cap X_j)$$

Since Y_i is irreducible, $Y_i = Y_i \cap X_j$ for some j. Similarly every X_j is contained in some Y_k. If $Y_i \subseteq X_j \subseteq Y_k$ then $Y_i = X_j = Y_k$ since we are assuming there are no inclusion relations between the Y_i. From this, we see that the decomposition is unique. □