Homework 8 Solutions

Dedekind Domains

We recall that a field k is *perfect* if every finite extension is separable. For example, finite fields, algebraically closed fields and fields of characteristic zero are all perfect. The field $\mathbb{F}_p(X)$ of fractions of the polynomial ring $\mathbb{F}_p[X]$ is not perfect, since the extension $\mathbb{F}_p(X^{1/p})/\mathbb{F}_p(X)$ is inseparable.

Problem 1. Let E/F be a finite separable field extension. Let A be a Dedekind domain with field of fractions F and let B be the integral closure of A in E. Thus B is a Dedekind domain. Assume that $B = A[\alpha]$ for some $\alpha \in B$. Let $\mathfrak{p}$ be a maximal ideal of A. Assume that $\overline{A} = A/\mathfrak{p}$ is a perfect field. We know that $\mathfrak{p}B$ can be factored into maximal ideals:

$$\mathfrak{p}B = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}.$$

The quotient $\overline{B}_i = B/\mathfrak{P}_i$ is an extension of $\overline{A}$. Let $n = [E : F]$ and let $f_i = \left[\overline{B}_i, \overline{A}\right]$. Prove that

$$\sum_{i=1}^{r} f_i \leqslant n.$$

Hint: Let Ω be an algebraic closure of $A/\mathfrak{p}$. Count the homomorphisms $B \longrightarrow \Omega$ extending the canonical map $A \longrightarrow A/\mathfrak{p}$. You may need the assumption that $\overline{A} = A/\mathfrak{p}$ to do the counting correctly.

Remark: The statement is true without the simplifying assumption $B = A[\alpha]$. In fact, the correct statement is

$$\sum_{i=1}^{r} e_i f_i = n.$$

See Lang, *Algebraic Number Theory*, Proposition 21 in Chapter I (page 24). This result is basic in algebraic number theory, and also in the study of algebraic curves. The numbers f_i and e_i are called the *residue class degree* and *ramification index*. Ramification is rare in the sense that all $e_i = 1$ for all but finitely many primes $\mathfrak{p}$ of A. Indeed, assuming $B = A[\alpha]$ it is easy to see that all $e_i = 1$ unless $\mathfrak{p}$ divides the discriminant of the irreducible polynomial satisfied by α.

Solution. Let $f(X) \in F[X]$ be the monic irreducible polynomial satisfied by α, which by Problem 2 of Homework 1 is in $A[X]$. Let $\overline{f}$ be the image of f in $\overline{A}[X]$. Let Ω be the algebraic closure of $\overline{A}$ and let $\phi : A \longrightarrow \Omega$ be the canonical map $A \longrightarrow A/\mathfrak{p}$ composed with the inclusion in Ω. We will count the ways of extending ϕ to a homomorphism $\Phi : B \longrightarrow \Omega$. The polynomial $\overline{f}$ may be reducible. It has at most n distinct roots, and $\Phi(\alpha)$ must be one of these, so there are $\leqslant n$ such extensions.

We know from previous homeworks that the primes in the factorization of $\mathfrak{p}B$ are exactly the primes $\mathfrak{P}_i$ of B above $\mathfrak{p}$. Thus the kernel of Φ must be one of these primes. The image $\Phi(\alpha)$ is a root of one of the irreducible factors of $\overline{f}$ in Ω. It must lie in the extension $B/\mathfrak{P}_i$ of $\overline{A}$. Since $\overline{A}$ is a perfect field, the extension $B/\mathfrak{P}_i$ is separable over $\overline{A}$, and it has f_i different embeddings in Ω over $\overline{A}$. Composing these with the canonical map $B \longrightarrow B/\mathfrak{P}_i$ gives f_i distinct homomorphisms $B \longrightarrow \Omega$ extending ϕ, proving $\sum f_i \leqslant n$.

Group Representations

Problem 2. Let

$$G = \langle x, y | x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle$$

be the nonabelian group of order 21. The cyclic group $N = \langle x \rangle$ has 7 linear characters. Compute the induced character $\mathrm{Ind}_N^G(\chi)$ for each of these.

Solution. We will make use of the character formula

$$\chi^G(g) = \sum_{t \in N \backslash G} \dot{\chi}(tgt^{-1})$$

where $\dot{\chi}$ is χ extended by zero off N to a function on G, and the summation is over a set of representatives of the left cosets Nx. Let $P = \langle y \rangle$ be the 3-Sylow subgroup. We may choose the representatives x to be the elements of N. Since N is normal, the induced character will vanish off N because $\dot{\chi}$ does. So

$$\chi^G(g) = \dot{\chi}(g) + \dot{\chi}(ygy^{-1}) + \dot{\chi}(y^2gy^{-2}). \tag{1}$$

For reference, here is the character table of G from Homework 7. Here ζ is a primitive 7-th root of unity, $\alpha = \zeta + \zeta^2 + \zeta^4$ and $\beta = \zeta^{-1} + \zeta^{-2} + \zeta^{-4}$. Note that α and β are complex conjugates.

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^{-1}
χ_3	1	1	1	ρ^{-1}	ρ
χ_4	3	α	β	0	0
χ_5	3	β	α	0	0

The value of χ is determined by $\chi(x)$ since χ is a homomorphism $N \to \mathbb{C}^\times$ and x generates N. If $\chi(x) = 1$ (the trivial character) then using (1) we obtain the following values.

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
1^G	3	3	3	0	0

and in this case we see that $1^G = \chi_1 + \chi_2 + \chi_3$. Next suppose that $\chi(x) = \zeta$. Then (1) gives

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
χ_4	3	α	β	0	0

so in this case $\chi^G = \chi_4$. We also get χ_4 if $\chi(x) = \zeta^2$ or ζ^4 and in the remaining cases, $\chi(x) = \zeta^{-1}, \zeta^{-2}$ or ζ^{-4} we find that $\chi^G = \chi_5$.

Problem 3. Let (π, V) be an irreducible representation of the finite group G, with character χ. Let $\mathcal{C}$ be a conjugacy class of G and let $g \in \mathcal{C}$. Prove that

$$|\mathcal{C}|\frac{\chi(g)}{\chi(1)}$$

is an algebraic integer, i.e. and element of $\overline{\mathbb{Q}}$ that is integral over $\mathbb{Z}$.

Hint: Let $\mathcal{C}_1, \cdots, \mathcal{C}_h$ be the conjugacy classes of G, and let g_i be a representative of each $\mathcal{C}_i$. Let $\mathfrak{C}_i = \sum_{g \in \mathcal{C}_i} g$; it is easy to see that these elements span the center of $\mathbb{C}[G]$. Deduce that the quantities $|\mathcal{C}_i|\chi(g_i)/\chi(1)$ span a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$.

Solution. The $\mathfrak{C}_i$ are central since conjugation by an element of G just permutes the summands $g \in \mathcal{C}_i$. They are elements of $\mathbb{Z}[G]$, and are a $\mathbb{C}$-basis of $Z(\mathbb{C}[G])$ since any element of the center must be of the form $\sum a_g \cdot g$ where a_g is constant on conjugacy classes, which implies that $\sum a_g \cdot g$ decomposes as a linear combination of the $\mathfrak{C}_i$. Now $\mathfrak{C}_i\mathfrak{C}_j \in \mathbb{Z}[G]$ has $\mathbb{Z}$-coefficients so when we express it as a sum $\mathfrak{C}_i\mathfrak{C}_j = \sum_k a_{ijk}\mathfrak{C}_k$ the coefficients a_{ijk} are (nonnegative) integers.

Now by Problem 6 of Homework 7 there exists $\mathbb{C}$-algebra homomorphism $\omega_\pi : Z(\mathbb{C}[G]) \longrightarrow \mathbb{C}$ such that if $\xi \in Z(\mathbb{C}[G])$ then ξ acts by the scalar $\omega_\pi(\xi)$ on the module V. Applying this to $\mathfrak{C}_i$ we see that $\omega_\pi(\mathfrak{C}_i)$ are complex numbers satisfying

$$\omega_\pi(\mathfrak{C}_i)\omega_\pi(\mathfrak{C}_j) = \sum_k a_{ijk}\omega_\pi(\mathfrak{C}_k), \qquad a_{ijk} \in \mathbb{Z}.$$

Their $\mathbb{Z}$-span is a finitely generated $\mathbb{Z}$-module that is a faithful $Z[\omega_\pi(\mathfrak{C}_i)]$-module and so the $\omega_\pi(\mathfrak{C}_i)$ are integral over $\mathbb{Z}$. By Problem 6 of Homework 7 the advertised values $|\mathcal{C}|\frac{\chi(g)}{\chi(1)}$ are these numbers.

Problem 4. Let G be a finite group and H a subgroup of index two. Let (π, V) be an irreducible representation of H. Prove that either $\mathrm{Ind}_H^G(\pi)$ is irreducible, or π can be extended to an irreducible representation of G. And in the second case, show that there are two such extensions.

Solution. Let χ be the character of H and χ^G the induced character of G. By Frobenius reciprocity

$$\langle \chi^G, \chi^G \rangle_G = \langle \chi^G, \chi \rangle_H.$$

This is the multiplicity of π in π^G when it is restricted back to H. In other words, the induced module V^G is a G-module, but we can restrict it back to H and decompose it into irreducible H-modules, and if we write

$$V^G = \bigoplus d_i U_i$$

where U_i are the distinct irreducibles, one of the U_i is V, and $d_i = \langle \chi^G, \chi \rangle_H$, but Schur orthogonality. Now $\dim(V^G) = 2\dim(V)$ and so $d_i \leqslant 2$.

Since $\langle \chi^G, \chi^G \rangle$ is thus seen to be $\leqslant 2$, either χ^G is irreducible or the inner product is 2. Decompose V^G into irreducible G-modules:

$$V^G = \bigoplus c_i V_i.$$

This means that $\sum c_i^2 = \langle \chi^G, \chi^G \rangle_G = 2$ so two of the c_i are 1, and the others are zero. Suppose that $c_1 = c_2 = 1$ and

$$V^G = V_1 \oplus V_2.$$

Let ψ_1 and ψ_2 be the characters of V_1 and V_2. Then $c_i = \langle \chi^G, \psi_i \rangle_G = 1$ for $i = 1, 2$ and by Frobenius reciprocity $\langle \chi, \psi_i \rangle_H = 1$, meaning that V_i, when restricted to H, contains a copy of V. Therefore $\dim(V_i) \geqslant \dim(V)$. But $\dim(V^G) = 2\dim(V)$ so $\dim(V_i) = \dim(V)$.

Now if V_i is restricted back to H, it must coincide with V because it contains a copy of V but has the same dimension. So we may identify $V_i = V$ as vector spaces, and what we have shown is that there are two extensions of the H-module structure of V to G.

Problem 5. Let G be a finite group and let V be an irreducible $\mathbb{C}[G]$-module, with character χ. Define the *Frobenius-Schur indicator*

$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Prove that $\varepsilon(\chi) = 0$ unless χ is real-valued, in which case it equals ± 1. Then $\varepsilon(\chi) = 1$ if the symmetric square module has a nonzero invariant vector (i.e. $\dim(\vee^2 V)^G = 1$) and $\varepsilon(\chi) = -1$ if $\dim(\wedge V)^G = 1$.

Solution. This is a continuation of Problem 5 in Homework 7. The tensor square module $\otimes^2 V$ decomposes as a direct sum of the symmetric square and exterior square modules

$\vee^2 V$ and $\wedge^2 V$, which may be realized as the $+1$ and -1 eigenspaces of the endomorphism $x \otimes y \mapsto y \otimes x$ of $\otimes^2 V$. The characters of these were computed in the quoted problem as:

$\otimes^2 V$	$\vee^2 V$	$\wedge^2 V$
$\chi(g)^2$	$\frac{1}{2}(\chi(g)^2 + \chi(g^2))$	$\frac{1}{2}(\chi(g)^2 - \chi(g^2))$

Now let us argue that the space of invariants $(\otimes^2 V)^G$ is at most one dimensional, and indeed is one-dimensional if and only if $\chi(g)$ is real-valued. Indeed, using Problem 3 of Homework 6, the dimension of the space of invariants is

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)^2 = \langle \chi, \overline{\chi} \rangle . \tag{2}$$

Now χ and $\overline{\chi}$ are both irreducible characters, so by Schur orthogonality

$$\dim\, (\otimes^2 V)^G = \begin{cases} 1 & \text{if } \chi = \overline{\chi}, \\ 0 & \text{otherwise.} \end{cases}$$

Now $(\otimes^2 V)^G$ is at most one dimensional, and it decomposes as

$$(\vee^2 V)^G \ \oplus \ (\wedge^2 V)^G.$$

If $\chi \neq \overline{\chi}$ both of these are subspaces of a 0-dimensional vector space, so they are also zero. Thus

$$\frac{1}{|G|} \sum_{g \in G} \frac{1}{2}(\chi(g)^2 + \chi(g^2)) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{2}(\chi(g)^2 - \chi(g^2)) = 0.$$

Subtracting these two equations gives $\varepsilon(\chi) = 0$. On the other hand if $\chi = \overline{\chi}$ is a real character then $(\vee^2 V)^G \ \oplus \ (\wedge^2 V)^G$ is one-dimensional, so one of these spaces is one-dimensional, the other zero dimensional. Suppose for example that $\dim\, (\vee^2 V)^G = 1$. Then by another application of Problem 3 of Homework 6 we get

$$1 = \frac{1}{|G|} \sum_{g \in G} \frac{1}{2}(\chi(g)^2 + \chi(g^2))$$

and we also know by (2) that

$$1 = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2.$$

Therefore $\varepsilon(\chi) = 1$. The case where $\dim\, (\vee^2 V)^G = -1$ is similar.

Mackey theory is concerned with intertwining operators (i.e. $\mathbb{C}[G]$-module homomorphisms) between induced representations. It is a powerful tool, and often exactly what is needed for some problem. Here is one version.

Theorem 1 (Mackey)**.** *Let G be a finite group and let H_1, H_2 be subgroups. Let (π_1, V_1) and (π_2, V_2) be representations of H_1, H_2 respectively. If $\gamma \in G$ and let $H_\gamma = H_2 \cap \gamma H_1 \gamma^{-1}$. Define a representation π_γ of H_γ by*

$$\pi_\gamma(h) = \pi_1(\gamma^{-1} h \gamma).$$

Then as vector spaces

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_{H_1}^G(\pi_1), \mathrm{Ind}_{H_2}^G(\pi_2)) \cong \bigoplus_{\gamma \in H_2 \backslash G / H_1} \mathrm{Hom}_{\mathbb{C}[H_\gamma]}(\pi_\gamma, \pi_2).$$

The notation means that we take a sum over a set of representatives γ for the set of double cosets $H_2 \gamma H_1$.

Proof. We will prove this in the lectures, or see Lang's *Algebra*, Theorem 7.7 on page 695. □

Problem 6. Let F be a finite field. Let $G = \mathrm{GL}(2, F)$ and let B be the subgroup

$$B = \left\{ \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} | y_1, y_2 \in F^\times, x \in F \right\}.$$

Let χ_1, χ_2 be linear characters of $F^\times$. Define a linear character χ of B by

$$\chi \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} = \chi_1(y_1)\chi_2(y_2) \tag{3}$$

and let $\pi(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi)$. Now let ψ_1 and ψ_2 be two more linear characters of $F^\times$.

$$\dim\ \mathrm{Hom}(\pi(\chi_1, \chi_2), \pi(\psi_1, \psi_2)) = A + B$$

where

$$A = \begin{cases} 1 & \text{if } \chi_1 = \psi_1, \chi_2 = \psi_2, \\ 0 & \text{otherwise,} \end{cases} \qquad B = \begin{cases} 1 & \text{if } \chi_1 = \psi_2, \chi_2 = \psi_1, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that $\pi(\chi_1, \chi_2)$ is irreducible if $\chi_1 \neq \chi_2$ and isomorphic to $\pi(\chi_2, \chi_1)$.

This problem shows how powerful Mackey's theorem is with a typical application. Problem 5 constructs about half the irreducible representations of $\mathrm{GL}(2, F)$.

Solution: We wish to apply Mackey's theorem, so we need to determine the double cosets $B \backslash G / B$. There are 2, since if $g \in G$ is not in $B = B \cdot 1_G \cdot B$ then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{with } c \neq 0$$

and then with

$$w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

we have

$$g = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} w \begin{pmatrix} c & d \\ & -c^{-1}\Delta \end{pmatrix}, \qquad \Delta = ad - bc$$

showing that $g \in BwB$. We may thus take the double coset representatives in Mackey's theorem to be $\gamma \in \{1_G, w\}$.

Now the subgroups $H_\gamma = B \cap \gamma B \gamma^{-1}$ are

$$H_{1_G} = B, \qquad H_w = T,$$

where

$$T = \left\{ \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} | y_1, y_2 \in F^\times \right\}.$$

Now we turn to Mackey's theorem. We must compute $\mathrm{Hom}_{\mathbb{C}[H_\gamma]}(\pi_\gamma, \pi_2)$ where π_2 is the linear character

$$\psi \begin{pmatrix} y_1 & * \\ 0 & y_2 \end{pmatrix} = \psi_1(y_1)\psi_2(y_2)$$

and π_γ is χ defined by (3) if $\gamma = 1$, and

$$\pi_\gamma \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} = {}^w\chi \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} := \chi_2(y_1)\chi_1(y_2)$$

if $\gamma = w$. The H_γ modules corresponding to the linear characters π_γ and π_2 are one-dimensionsional, so they are irreducible, and the Hom space is nonzero if and only if they are equal. So

$$\mathrm{Hom}_{\mathbb{C}[H_{1_G}]}(\pi_{1_G}, \pi_2) = \mathrm{Hom}_{\mathbb{C}[B]}(\chi, \psi) = \begin{cases} 1 & \text{if } \chi_1 = \psi_1, \chi_2 = \psi_2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Hom}_{\mathbb{C}[H_w]}(\pi_w, \pi_2) = \mathrm{Hom}_{\mathbb{C}[T]}({}^w\chi, \psi) = \begin{cases} 1 & \text{if } \chi_1 = \psi_2, \chi_2 = \psi_1, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the first statement. For the second, assume that $\chi_1 \neq \chi_2$ and compute dim $\mathrm{Hom}_{\mathbb{C}[G]}(\pi(\chi_1, \chi_2), \pi$
Then $A = 1$ and $B = 0$ so

$$\dim\ \mathrm{Hom}_{\mathbb{C}[G]}(\pi(\chi_1, \chi_2), \pi(\chi_1, \chi_2)) = 1.$$

This implies that $\pi(\chi_1, \chi_2)$ is irreducible. Also we may compute

$$\dim\ \mathrm{Hom}_{\mathbb{C}[G]}(\pi(\chi_1, \chi_2), \pi(\chi_2, \chi_1)) = 1$$

since for this calculation $A = 0$ and $B = 1$. This implies that the irreducible representations $\pi(\chi_1, \chi_2)$ and $\pi(\chi_2, \chi_1)$ are nonisomorphic.