Homework 8

Dedekind Domains

We recall that a field k is *perfect* if every finite extension is separable. For example, finite fields, algebraically closed fields and fields of characteristic zero are all perfect. The field $\mathbb{F}_p(X)$ of fractions of the polynomial ring $\mathbb{F}_p[X]$ is not perfect, since the extension $\mathbb{F}_p(X^{1/p})/\mathbb{F}_p(X)$ is inseparable.

Problem 1. Let E/F be a finite separable field extension. Let A be a Dedekind domain with field of fractions F and let B be the integral closure of A in E. Thus B is a Dedekind domain. Assume that $B = A[\alpha]$ for some $\alpha \in B$. Let $\mathfrak{p}$ be a maximal ideal of A. Assume that $\overline{A} = A/\mathfrak{p}$ is a perfect field. We know that $\mathfrak{p}B$ can be factored into maximal ideals:

$$\mathfrak{p}B = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}.$$

The quotient $\overline{B}_i = B/\mathfrak{P}_i$ is an extension of $\overline{A}$. Let $n = [E : F]$ and let $f_i = [\overline{B}_i : \overline{A}]$. Prove that

$$\sum_{i=1}^{r} f_i \leqslant n.$$

Hint: Let Ω be an algebraic closure of $A/\mathfrak{p}$. Count the homomorphisms $B \longrightarrow \Omega$ extending the canonical map $A \longrightarrow A/\mathfrak{p}$. You may need the assumption that $\overline{A} = A/\mathfrak{p}$ to do the counting correctly.

Remark: The statement is true without the simplifying assumption $B = A[\alpha]$. In fact, the correct statement is

$$\sum_{i=1}^{r} e_i f_i = n.$$

See Lang, *Algebraic Number Theory*, Proposition 21 in Chapter I (page 24). This result is basic in algebraic number theory, and also in the study of algebraic curves. The numbers f_i and e_i are called the *residue class degree* and *ramification index*. Ramification is rare in the sense that all $e_i = 1$ for all but finitely many primes $\mathfrak{p}$ of A. Indeed, assuming $B = A[\alpha]$ it is easy to see that all $e_i = 1$ unless $\mathfrak{p}$ divides the discriminant of the irreducible polynomial satisfied by α.

Group Representations

Problem 2. Let

$$G = \langle x, y | x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle$$

be the nonabelian group of order 21. The cyclic group $N = \langle x \rangle$ has 7 linear characters. Compute the induced character $\mathrm{Ind}_N^G(\chi)$ for each of these.

Problem 3. Let (π, V) be an irreducible representation of the finite group G, with character χ. Let $\mathcal{C}$ be a conjugacy class of G and let $g \in \mathcal{C}$. Prove that

$$|\mathcal{C}| \frac{\chi(g)}{\chi(1)}$$

is an algebraic integer, i.e. and element of $\overline{\mathbb{Q}}$ that is integral over $\mathbb{Z}$.

Hint: Let $\mathcal{C}_1, \cdots, \mathcal{C}_h$ be the conjugacy classes of G, and let g_i be a representative of each $\mathcal{C}_i$. Let $\mathfrak{C}_i = \sum_{g \in \mathcal{C}_i} g$; it is easy to see that these elements span the center of $\mathbb{C}[G]$. Deduce that the quantities $|\mathcal{C}_i|\chi(g_i)/\chi(1)$ span a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$.

Problem 4. Let G be a finite group and H a subgroup of index two. Let (π, V) be an irreducible representation of H. Prove that either $\mathrm{Ind}_H^G(\pi)$ is irreducible, or π can be extended to an irreducible representation of G. And in the second case, show that there are two such extensions.

Problem 5. Let G be a finite group and let V be an irreducible $\mathbb{C}[G]$-module, with character χ. Define the *Frobenius-Schur indicator*

$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Prove that $\varepsilon(\chi) = 0$ unless χ is real-valued, in which case it equals ± 1. Then $\varepsilon(\chi) = 1$ if the symmetric square module has a nonzero invariant vector (i.e. $\dim(\vee^2 V)^G = 1$) and $\varepsilon(\chi) = -1$ if $\dim(\wedge^2 V)^G = 1$.

Mackey theory is concerned with intertwining operators (i.e. $\mathbb{C}[G]$-module homomorphisms) between induced representations. It is a powerful tool, and often exactly what is needed for some problem. Here is one version.

Theorem 1 (Mackey)**.** *Let G be a finite group and let H_1, H_2 be subgroups. Let (π_1, V_1) and (π_2, V_2) be representations of H_1, H_2 respectively. If $\gamma \in G$ and let $H_\gamma = H_2 \cap \gamma H_1 \gamma^{-1}$. Define a representation π_γ of H_γ by*

$$\pi_\gamma(h) = \pi_1(\gamma^{-1} h \gamma).$$

Then as vector spaces

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_{H_1}^G(\pi_1), \mathrm{Ind}_{H_2}^G(\pi_2)) \cong \bigoplus_{\gamma \in H_2 \backslash G / H_1} \mathrm{Hom}_{\mathbb{C}[H_\gamma]}(\pi_\gamma, \pi_2).$$

The notation means that we take a sum over a set of representatives γ for the set of double cosets $H_2 \gamma H_1$.

Proof. We will prove this in the lectures, or see Lang's *Algebra*, Theorem 7.7 on page 695. □

Problem 6. Let F be a finite field. Let $G = \mathrm{GL}(2, F)$ and let B be the subgroup

$$B = \left\{ \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} | y_1, y_2 \in F^\times, x \in F \right\}.$$

Let χ_1, χ_2 be characters of $F^\times$. Define a linear character χ of B by

$$\chi \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} = \chi_1(y_1)\chi_2(y_2) \tag{1}$$

and let $\pi(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi)$. Now let ψ_1 and ψ_2 be two more characters of $F^\times$. Prove that

$$\dim\ \mathrm{Hom}(\pi(\chi_1, \chi_2), \pi(\psi_1, \psi_2)) = A + B$$

where

$$A = \begin{cases} 1 & \text{if } \chi_1 = \psi_1, \chi_2 = \psi_2, \\ 0 & \text{otherwise,} \end{cases} \qquad B = \begin{cases} 1 & \text{if } \chi_1 = \psi_2, \chi_2 = \psi_1, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that $\pi(\chi_1, \chi_2)$ is irreducible if $\chi_1 \neq \chi_2$ and isomorphic to $\pi(\chi_2, \chi_1)$.

This problem shows how powerful Mackey's theorem is with a typical application. Problem 5 constructs about half the irreducible representations of $\mathrm{GL}(2, F)$.