Homework 7 Solutions

Dedekind Domains

Problem 1. Let A be a Dedekind domain and let $\mathfrak{a}, \mathfrak{b}$ be nonzero ideals. We say that $\mathfrak{a}$ *divides* $\mathfrak{b}$ and write $\mathfrak{a}|\mathfrak{b}$ if $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ for some ideal $\mathfrak{c}$. Show that $\mathfrak{a}|\mathfrak{b}$ if and only if $\mathfrak{a} \supseteq \mathfrak{b}$.

Solution. If $\mathfrak{a}|\mathfrak{b}$ any element of $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ is clearly in $\mathfrak{a}$, so $\mathfrak{a} \supseteq \mathfrak{b}$. Conversely, if $\mathfrak{a} \supseteq \mathfrak{b}$ let

$$\mathfrak{a}^{-1} = \{x \in F | x\mathfrak{a} \subseteq A\}.$$

Clearly $\mathfrak{a}^{-1} \subseteq \mathfrak{b}^{-1}$ so if we define $\mathfrak{c}$ to be the fractional ideal $\mathfrak{a}^{-1}\mathfrak{b}$ then $\mathfrak{c} \subseteq \mathfrak{b}^{-1}\mathfrak{b} = A$ by the invertibility of fractional ideals (Homework 4, Problem 4). Thus $\mathfrak{c}$ is an ideal, and $\mathfrak{b} = \mathfrak{a}\mathfrak{a}^{-1}\mathfrak{b} = \mathfrak{a}\mathfrak{c}$, proving that $\mathfrak{a}|\mathfrak{b}$.

Problem 2. Let K/F be a finite separable extension, and let A be a Dedekind domain whose field of fractions is F. Let B be the integral closure of A in K. By Problem 2 in Homework 4, B is also a Dedekind domain. Let $\mathfrak{P}$ be a maximal ideal of B. Show that $\mathfrak{P} \cap A$ is a nonzero prime ideal of A. Furthermore if $\mathfrak{p}$ is a maximal ideal of A, prove that $\mathfrak{P} \cap A = \mathfrak{p}$ if and only if $\mathfrak{P}|\mathfrak{p}B$.

Solution. To prove that $\mathfrak{P} \cap A$ is nonzero, let $0 \neq x \in \mathfrak{P}$. Let E be the normal closure of K, and let C be the integral closure of A in E. Then by the formula at the top of Page 285 of Lang's *Algebra* the norm

$$N(x) = \prod \sigma(x)$$

where the product is over the set $\mathcal{E}$ of embeddings $\sigma : K \longrightarrow E$ over F. Let $\sigma_0 \in \mathcal{E}$ be the embedding that maps x to itself. Then

$$\frac{N(x)}{x} = \prod_{\sigma \neq \sigma_0} \sigma(x)$$

is integral over A. Moreover $N(x) \in F$ while $x \in K$ so $N(x)/x \in K$. Therefore $N(x)/x \in B$ since B is the integral closure of A in K. It follows that $N(x) = (N(x)/x) \cdot x \in \mathfrak{P}$ is a nonzero element of $A \cap \mathfrak{P}$.

Let us prove the divisibility assertions. First suppose that $\mathfrak{P} \cap A = \mathfrak{p}$. Then

$$\mathfrak{p}B = (\mathfrak{P} \cap A)B \subseteq \mathfrak{P}B = \mathfrak{P}$$

so $\mathfrak{P}|\mathfrak{p}B$ by Problem 1. Conversely assume that $\mathfrak{P}|\mathfrak{p}B$. By Problem 1, $\mathfrak{p} \subset \mathfrak{p}B \subseteq \mathfrak{P}$. Thus $\mathfrak{p} \subseteq \mathfrak{P} \cap A$. Since $\mathfrak{p}$ is maximal, it follows that $\mathfrak{P} \cap A = \mathfrak{p}$.

Group representations

Let G be a group. Recall that the *commutator subgroup* or *derived group* G' is the subgroup generated by commutators $[x, y] = xyx^{-1}y^{-1}$. As we discussed in class, it is a normal subgroup and G/G' is abelian. Moreover any homomorphism $G \longrightarrow A$, where A is an abelian group, factors uniquely through the quotient G/G'.

Also recall that the characters of the one-dimensional representations of G are called *linear characters*. It is easy to see that these are just the homomorphisms $G \longrightarrow \mathbb{C}^\times$. Thus every linear character factors through G/G'.

Problem 3. Let G be a nonabelian group of order 21 with presentation

$$G = \langle x, y | x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle.$$

Determine the conjugacy classes and give a representative g_i for each. To describe a character χ of G it is sufficient to tell us $\chi(g_i)$ for each conjugacy class representatives. Let $Q = \langle x \rangle$ be the 7-Sylow subgroup, which is normal. Show that $Q = G'$ and determine the linear characters of G. Use this information to determine the total number of irreducible representations and their degrees.

Solution. The conjugacy classes are $\{1\}$, $\{x, x^2, x^4\}$, $\{x^{-1}, x^{-2}, x^{-4}\}$, $\{yx^i | 0 \leqslant i < 7\}$ and $\{y^2x^i | 0 \leqslant i < 7\}$. There are 5 conjugacy classes, so there are 5 irreducible representations.

By the Sylow theorem, Q is normal, and $G/Q \cong Z_3$ is abelian, so G' is contained in Q. On the other hand $xyx^{-1}y^{-1} = x^{-1}$ so $G' \supseteq \langle x \rangle = Q$. Thus $G' = Q$.

Since $G/G' \cong Z_3$ there are 3 linear characters. Thus if $d_1, \cdots, d_5$ are the character degrees we have $d_1 = d_2 = d_3 = 1$ and $\sum d_i^2 = 21$. The only possibility is that $d_2 = d_3 = 3$.

Problem 4. Continuing from Problem 3, note that if ζ is a 7-th root of unity and

$$\xi = \begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & \zeta^4 \end{pmatrix}, \qquad \eta = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

then $\xi^7 = \eta^3 = I$ (the identity matrix) and $\eta\xi\eta^{-1} = \xi^2$. Use this information to construct an irreducible representation of G and finish computing the character table.

Solution. The linear characters are the characters of $G/Q \cong Z_3$ which may be pulled back to G under the canonical homomorphism $G \longrightarrow G/Q$. If $\rho = e^{2\pi i/3}$ we obtain 3 linear characters:

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^{-1}
χ_3	1	1	1	ρ^{-1}	ρ

Now since ξ and η satisfy the defining relations of G there is a homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$ such that $\pi(x) = \xi$ and $\pi(y) = \eta$. Let $\alpha = \zeta + \zeta^2 + \zeta^4$ and $\beta = \overline{\alpha} = \zeta^{-1} + \zeta^{-2} + \zeta^{-4}$.

We see that the character χ of π satisfies $\chi(x) = \alpha$ and $\chi(x^{-1}) = \beta$. Furthermore $\mathrm{tr}(\eta\xi^i) = 0$ so this character has known values. Let us check that χ is irreducible. We have

$$\langle \chi, \chi \rangle = \frac{1}{21}(\chi(1)^2 + 3|\chi(x)|^2 + 3|\chi(x^{-1})|^2) = \frac{1}{21}(9 + 6\alpha\beta).$$

Note that $\alpha\beta = 3 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 = 2$ so $\langle \chi, \chi \rangle = 1$, proving that χ is irreducible. We obtain another degree 3 representation on replacing ζ by ζ^{-1}, and we may now complete the character table:

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^{-1}
χ_3	1	1	1	ρ^{-1}	ρ
χ_4	3	α	β	0	0
χ_5	3	β	α	0	0

If $\mathcal{V}$ is the category of finite dimensional vector spaces and $\mathcal{F}$ is a functor from $\mathcal{V}$ to itself, we may apply $\mathcal{F}$ to a representation $\pi : G \longrightarrow \mathrm{GL}(V)$: defining

$$(\mathcal{F}\pi)(g) = \mathcal{F}(\pi(g)) : \mathcal{F}V \longrightarrow \mathcal{F}V$$

gives a representation $\mathcal{F}\pi : G \longrightarrow \mathrm{GL}(\mathcal{F}V)$. In the next exercise we consider the functors of tensor square, exterior square and symmetric square, which we will denote $\otimes^2$, $\wedge^2$ and $\vee^2$.

Problem 5. Let V be a $\mathbb{C}[G]$-module, and let $\pi : G \longrightarrow \mathrm{GL}(V)$ be a representation. Let $\chi : G \longrightarrow \mathbb{C}$ be the character of π. Show that the characters of $\otimes^2\pi$, $\wedge^2\pi$ and $\vee^2\pi$ are

$$\otimes^2\chi(g) = \chi(g)^2, \qquad \wedge^2\chi(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)), \qquad \vee^2\chi(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)).$$

Hint: Express these in terms of the eigenvalues of $\pi(g)$.

Solution: Let $v_2, \cdots, v_d$ be the eigenvectors of $\pi(g)$, and ε_i the corresponding eigenvalues. Then $v_i \otimes v_j$ are a basis of $\otimes^2 V$ and the corresponding eigenvalues of $\pi(g) \otimes \pi(g)$ are $\varepsilon_i\varepsilon_j$. Thus

$$\otimes^2\chi(g) = \sum_{i,j=1}^{d} \varepsilon_i\varepsilon_j = \chi(g)^2.$$

The $v_i \wedge v_j$ with $i < j$ are the eigenvalues of $\wedge^2\pi(g)$ on $V \wedge V$ and so

$$\wedge^2\chi(g) = \sum_{i<j} \varepsilon_i\varepsilon_j.$$

The $v_i \vee v_j$ with $i \leqslant j$ are the eigenvalues of $\vee^2\pi(g)$ on $V \vee V$ and so

$$\vee^2\chi(g) = \sum_{i\leqslant j} \varepsilon_i\varepsilon_j.$$

Now the eigenvalues of $\pi(g^2)$ are ε_i^2 so

$$\chi(g^2) = \sum_i \varepsilon_i^2, \qquad \chi(g)^2 - \chi(g^2) = 2\sum_{i<j} \varepsilon_i \epsilon_j,$$

$$\chi(g^2) + \chi(g^2) = 2\sum_{i\leqslant j} \varepsilon_i \varepsilon_j$$

and the result follows.

Problem 6. Let $\pi : G \longrightarrow \mathrm{GL}(V)$ be a finite-dimensional complex representation of the finite group G. Note that π extends to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \longrightarrow \mathrm{End}(V)$, also denoted π. Let $\mathcal{Z}$ be the center of $\mathbb{C}[G]$. Show that there exists a $\mathbb{C}$-algebra homomorphism $\omega_\pi : \mathcal{Z} \longrightarrow \mathbb{C}$ such that $\pi(\xi)v = \omega_\pi(\xi)v$ for $\xi \in \mathcal{Z}$, called the *central character* of π. Let $g \in G$ and let $\mathcal{C}$ be the conjugacy class of g. Let $\mathcal{C}$ be a conjugacy class of G. Observe that $\mathfrak{C} = \sum_{h\in\mathcal{C}} h \in \mathcal{Z}$. Prove that

$$\omega_\pi(\mathfrak{C}) = \frac{\chi(g)|\mathcal{C}|}{\chi(1)}.$$

where χ is the character of π.

Solution. First note that if $\xi \in \mathcal{Z}$ then $\pi(\xi)\pi(g) = \pi(g)\pi(\xi)$. By Schur's Lemma, $\pi(\xi)$ is a scalar endomorphism of V. Let $\omega_\pi(\xi)$ be its scalar eigenvalue, so

$$\pi(\xi)v = \omega_\pi(\xi)v.$$

If $\xi, \eta \in \mathcal{Z}$ then

$$\omega_\pi(\xi)\omega_\pi(\eta) = \pi(\xi)\pi(\eta)I_V = \pi(\xi\eta)I_V = \omega_\pi(\xi\eta)$$

and $\omega_\pi : \mathcal{Z} \longrightarrow \mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism.

Applying this to $\mathfrak{C} = \sum_{h\in\mathcal{C}} h$, which is evidently central, let $\lambda = \omega_\pi(\mathfrak{C})$. We have

$$\lambda v = \pi(\mathfrak{C})v = \sum_{g\in\mathcal{C}} \pi(h)v.$$

To compute λ, we take the trace. Since $\chi(1) = \dim(V)$

$$\chi(1)\lambda = \mathrm{tr}(\lambda I_V) = \sum_{g\in\mathcal{C}} \chi(g) = |\mathcal{C}|\chi(g)$$

snd so λ has the advertised value.