Homework 7

Dedekind Domains

Problem 1. Let A be a Dedekind domain and let $\mathfrak{a}, \mathfrak{b}$ be nonzero ideals. We say that $\mathfrak{a}$ *divides* $\mathfrak{b}$ and write $\mathfrak{a}|\mathfrak{b}$ if $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ for some ideal $\mathfrak{c}$. Show that $\mathfrak{a}|\mathfrak{b}$ if and only if $\mathfrak{a} \supseteq \mathfrak{b}$.

Problem 2. Let K/F be a finite separable extension, and let A be a Dedekind domain whose field of fractions is F. Let B be the integral closure of A in K. By Problem 2 in Homework 4, B is also a Dedekind domain. Let $\mathfrak{P}$ be a maximal ideal of B. Show that $\mathfrak{P} \cap A$ is a nonzero prime ideal of A. Furthermore if $\mathfrak{p}$ is a maximal ideal of A, prove that $\mathfrak{P} \cap A = \mathfrak{p}$ if and only if $\mathfrak{P}|\mathfrak{p}B$.

Group representations

Let G be a group. Recall that the *commutator subgroup* or *derived group* G' is the subgroup generated by commutators $[x, y] = xyx^{-1}y^{-1}$. As we discussed in class, it is a normal subgroup and G/G' is abelian. Moreover any homomorphism $G \longrightarrow A$, where A is an abelian group, factors uniquely through the quotient G/G'.

Also recall that the characters of the one-dimensional representations of G are called *linear characters*. It is easy to see that these are just the homomorphisms $G \longrightarrow \mathbb{C}^\times$. Thus every linear character factors through G/G'.

Problem 3. Let G be a nonabelian group of order 21 with presentation

$$G = \langle x, y | x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle.$$

Determine the conjugacy classes and give a representative g_i for each. To describe a character χ of G it is sufficient to tell us $\chi(g_i)$ for each conjugacy class representatives. Let $Q = \langle x \rangle$ be the 7-Sylow subgroup, which is normal. Show that $Q = G'$ and determine the linear characters of G. Use this information to determine the total number of irreducible representations and their degrees.

Problem 4. Continuing from Problem 3, note that if ζ is a 7-th root of unity and

$$\xi = \begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & \zeta^4 \end{pmatrix}, \qquad \eta = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

then $\xi^7 = \eta^3 = I$ (the identity matrix) and $\eta\xi\eta^{-1} = \xi^2$. Use this information to construct an irreducible representation of G and finish computing the character table.

If $\mathcal{V}$ is the category of finite dimensional vector spaces and $\mathcal{F}$ is a functor from $\mathcal{V}$ to itself, we may apply $\mathcal{F}$ to a representation $\pi : G \longrightarrow \mathrm{GL}(V)$: defining

$$(\mathcal{F}\pi)(g) = \mathcal{F}(\pi(g)) : \mathcal{F}V \longrightarrow \mathcal{F}V$$

gives a representation $\mathcal{F}\pi : G \longrightarrow \mathrm{GL}(\mathcal{F}V)$. In the next exercise we consider the functors of tensor square, exterior square and symmetric square, which we will denote $\otimes^2$, $\wedge^2$ and $\vee^2$.

Problem 5. Let V be a $\mathbb{C}[G]$-module, and let $\pi : G \longrightarrow \mathrm{GL}(V)$ be a representation. Let $\chi : G \longrightarrow \mathbb{C}$ be the character of π. Show that the characters of $\otimes^2\pi$, $\wedge^2\pi$ and $\vee^2\pi$ are

$$\otimes^2\chi(g) = \chi(g)^2, \qquad \wedge^2\chi(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)), \qquad \vee^2\chi(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)).$$

Hint: Express these in terms of the eigenvalues of $\pi(g)$.

Problem 6. Let $\pi : G \longrightarrow \mathrm{GL}(V)$ be an irreducible finite-dimensional complex representation of the finite group G. Note that π extends to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \longrightarrow \mathrm{End}(V)$, also denoted π. Let $\mathcal{Z}$ be the center of $\mathbb{C}[G]$. Show that there exists a $\mathbb{C}$-algebra homomorphism $\omega_\pi : \mathcal{Z} \longrightarrow \mathbb{C}$ such that $\pi(\xi)v = \omega_\pi(\xi)v$ for $\xi \in \mathcal{Z}$, called the *central character* of π. Let $g \in G$ and let $\mathcal{C}$ be the conjugacy class of g. Let $\mathcal{C}$ be a conjugacy class of G. Observe that $\mathfrak{C} = \sum_{h \in \mathcal{C}} h \in \mathcal{Z}$. Prove that

$$\omega_\pi(\mathfrak{C}) = \frac{\chi(g)|\mathcal{C}|}{\chi(1)}.$$

where χ is the character of π.