Homework 6

The goal of the first problem is to prove the Artin-Rees Lemma, which was stated without proof in *Dimension II* and applied to the Krull Dimension Theorem.

Problem 1. Let A be a Noetherian commutative ring, $\mathfrak{a}$ an ideal, and M a finitely-generated A-module. Let $R = A[x]$ be the polynomial ring and $M[x]$ the R-module of polynomials $\sum_i m_i x^i$ with $m_i \in M$.

$$R' = \left\{ \sum_{k=0}^{m} a_k x^k \in R | a_k \in \mathfrak{a}^k \right\}, \qquad M' = \left\{ \sum_{k=0}^{n} m_k x^k | m_k \in \mathfrak{a}^k M \right\}$$

(i) Prove that R' is a Noetherian ring and M' a finitely generated R'-module.

(ii) Let N be a submodule of M. Let $N' = \left\{ \sum_{k=0}^{n} m_k x^k | m_k \in N \cap \mathfrak{a}^k M \right\}$. Show that N' is a finitely generated R'-module, and deduce the Artin-Rees Lemma.

Now let A be a commutative ring and M an A-module. Define a topology on M in which a subset $U \subseteq M$ is open if and only if for every $x \in U$, the set U contains $x + \mathfrak{a}^n M$ for some n. Thus a sequence $x_i \longrightarrow 0$ if and only if for every n, $x_i \in \mathfrak{a}$ for sufficiently large i. This is called the $\mathfrak{a}$–adic topology on M. This leads to the $\mathfrak{a}$-*adic completion* $\hat{M}$ whose use is a powerful technique for studying M.

Problem 2. Assume that A is Noetherian, and let M a finitely generated $\mathfrak{a}$-module with the $\mathfrak{a}$-adic topology. Let N be a submodule of M. We have two topologies on N: its own $\mathfrak{a}$-adic topology, and its subspace topology as a submodule of M. Show that these two topologies are the same.

Hint: Use the Artin-Rees Lemma.

Representations

If M is a module for the group algebra $\mathbb{C}[G]$ of a finite group, and $g \in G$, we define an endomorphism $\pi(g) : M \longrightarrow M$ by $\pi(g)v = g \cdot v$. This is a representation of G. Conversely, given a representation $\pi : G \longrightarrow \mathrm{GL}(M)$, where M is a complex vector space, then M admits the structure of a $\mathbb{C}[G]$-module by

$$\left(\sum_{g \in G} a_g g \right) \cdot v = \sum a_g \pi(g)(v), \qquad v \in M.$$

In these notes we will always assume that the module M is *finite-dimensional*. The *character* of M is the character of the corresponding representation. So

$$\chi(g) = \text{tr } \pi(g).$$

If V is a vector space, a map $p : V \longrightarrow V$ is called a *projection* if $p^2 = p$.

Problem 3. Let G be a finite group, $R = \mathbb{C}[G]$ its group algebra. If M is an R-module, let

$$M^G = \{x \in M | gx = x \text{ for all } g \in G\}$$

be the module of invariants. Define the endomorphism $p : M \longrightarrow M$ by

$$p(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v. \tag{1}$$

Prove that p is a projection map whose image is M^G and deduce that

$$\dim(M^G) = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

Problem 4. Let M and N be $\mathbb{C}[G]$-modules. Let $W = \text{Hom}_{\mathbb{C}}(M, N)$ be the space of linear transformations $T : M \longrightarrow N$. Show that W has the structure of a $\mathbb{C}[G]$-module such that

$$(g\phi)(x) = g \cdot \phi(g^{-1} \cdot x), \qquad x \in M.$$

Show that the character of W is $\chi_M(g^{-1})\chi_N(g)$, where χ_M and χ_N are the characters of M and N.

Problem 5. Let F be a field of characteristic p, and let $G = \langle \sigma \rangle$ a cyclic group of order p generated by $\sigma \in G$. Let R be the group algebra $F[G]$. Show that R has elements $\tau_1, \cdots, \tau_p$ such that

$$\begin{aligned} \sigma\tau_1 &= \tau_1, \\ \sigma\tau_i &= \tau_i - \tau_{i-1} \quad (i > 1). \end{aligned}$$

Describe all ideals of R and deduce that R is not a semisimple ring.