Homework 5 Solutions

All rings are commutative with unit.

Recall that an ideal $\mathfrak{q}$ in a ring A is *primary* if $xy \in \mathfrak{q}$ implies that either $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some n. Equivalently, every zero divisor in $A/\mathfrak{q}$ is nilpotent.

Lemma 1. *If $\mathfrak{q}$ is primary then $r(\mathfrak{q})$ is prime.*

Proof. Suppose that $xy \in r(\mathfrak{q})$. Then $x^N y^N \in \mathfrak{q}$ for some N. Thus either $x^N \in \mathfrak{q}$ or $(y^N)^n \in \mathfrak{q}$ for some n. Thus either $x \in r(\mathfrak{q})$ or $y \in r(\mathfrak{q})$. □

If $\mathfrak{q}$ is primary and $r(\mathfrak{q}) = \mathfrak{p}$ then we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*.

Lemma 2. *Suppose that $\mathfrak{q}_1, \mathfrak{q}_2$ are $\mathfrak{p}$-primary ideals. Then $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is also $\mathfrak{p}$-primary.*

Proof. We have $r(\mathfrak{q}_1 \cap \mathfrak{q}_2) = r(\mathfrak{q}_1) \cap r(\mathfrak{q}_2) = \mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}$. To show $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is $\mathfrak{p}$-primary, suppose that $xy \in \mathfrak{q}_1 \cap \mathfrak{q}_2$ but $x \notin \mathfrak{q}_1 \cap \mathfrak{q}_2$. Then either $x \notin \mathfrak{q}_1$ or $x \notin \mathfrak{q}_2$. By symmetry, we may assume that $x \notin \mathfrak{q}_1$. Then since $xy \in \mathfrak{q}_1$ which is $\mathfrak{p}$-primary, we have $y \in \mathfrak{p} = r(\mathfrak{q}_1 \cap \mathfrak{q}_2)$. □

Theorem 3. *If A is Noetherian, then every ideal may be expressed as a finite intersection of primary ideals.*

Proof. We will prove this in class. See Lang's *Algebra*, Theorem X.3.3 on page 423. □

Write the ideal $\mathfrak{a}$ as an intersection of primary ideals $\mathfrak{q}_i$:

$$\mathfrak{a} = \bigcap_{i=1}^{d} \mathfrak{q}_i.$$

By Lemma 2 we may assume that the radicals $r(\mathfrak{q}_i)$ are all distinct. Moreover, we may obviously discard any $\mathfrak{q}_i$ if that does not change the intersection. Assuming this, the decomposition is called a (reduced) *primary decomposition*. The decomposition is not unique but has some uniqueness properties. The primary decomposition is discussed in Lang's *Algebra* Chapter 10, but see the book of Atiyah and Macdonald, Chapter 4, for more information. Atiyah and Macdonald is available on-line through the Stanford Libraries.

The notion of a primary ideal is needed in the dimension theory.

The following well-known fact may be useful in doing Problem 1. The set $\mathfrak{N}$ of nilpotent elements of a ring A is called the *nilradical*. It is an ideal, the radical of (0).

Proposition 4. *The nilradical is the intersection of all prime ideals of A.*

Proof. See Lang's *Algebra*, Corollary X.2.2 on page 417. □

Problem 1. Show that if $r(\mathfrak{a})$ is maximal, then $A/\mathfrak{a}$ is a local ring with a single prime ideal. The ideal $\mathfrak{a}$ is primary.

Solution: Let $\mathfrak{m} = r(\mathfrak{a})$. Let $\overline{A} = A/\mathfrak{a}$ and let $\overline{\mathfrak{m}}$ be the image of $\mathfrak{m}$. An element $\overline{x} \in \overline{A}$ is nilpotent if and only if $x^N \in \mathfrak{a}$ for some $\mathfrak{a}$, that is, if and only if $x \in \mathfrak{m}$. Thus $\overline{\mathfrak{m}}$ is the nilradical of $\overline{A}$. Since $\overline{\mathfrak{m}}$ is maximal, and the nilradical is the intersection of all prime ideals of $\overline{A}$, it follows that $\overline{\mathfrak{m}}$ is the unique prime ideal of $\overline{A}$. In particular it is the unique maximal ideal so $\overline{A}$ is local. Any zero divisor is an element of $\overline{\mathfrak{m}}$, hence is nilpotent, proving that $\mathfrak{a}$ is primary.

One might hope from Problem 1 that if $r(\mathfrak{a})$ is only assumed to be prime that $\mathfrak{a}$ might be primary. However the next problem shows that this is not true.

Problem 2. Let A be the quotient of the polynomial ring $\mathbb{C}[X, Y, Z]$ by the ideal $(X^2 - YZ)$ and let x, y, z be the images of X, Y, Z in A. Let $\mathfrak{p}$ be the ideal (x, z). Show that $\mathfrak{p}$ is prime and that the radical of $\mathfrak{p}^2$ is $\mathfrak{p}$, but that $\mathfrak{p}^2$ is not primary.

Solution. To show that $\mathfrak{p}$ is prime note that it is the image of the ideal (X, Z) of $\mathbb{C}[X, Y, Z]$ in A. Since (X, Z) is prime, $\mathfrak{p}$ is prime. Let us show that $r(\mathfrak{p}^2) = \mathfrak{p}$. Since both generators x and z of $\mathfrak{p}$ are in $\mathfrak{p}^2$ we have $\mathfrak{p} \subseteq r(\mathfrak{p}^2)$. On the other hand $\mathfrak{p}^2 \subseteq \mathfrak{p}$ so $r(\mathfrak{p}^2) \subseteq r(\mathfrak{p}) = \mathfrak{p}$ since $\mathfrak{p}$ is prime. Thus $r(\mathfrak{p}^2) = \mathfrak{p}$. We have $yz = x^2 \in \mathfrak{p}^2$, but $z \notin \mathfrak{p}^2$ and $y^N \notin \mathfrak{p}^2$ for any N. Hence $\mathfrak{p}^2$ is not primary.

Problem 3. Let A be the polynomial ring $k[x, y, z]$, where k is a field. Let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$ and $\mathfrak{m} = (x, y, z)$. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Prove that

$$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$$

is a primary decomposition.

Solution. The ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are primary since they are prime. The ideal $\mathfrak{m}^2$ is primary by Problem 1 since its radical $\mathfrak{m}$ is a maximal ideal. It remains to be shown that the stated decomposition is true. It is obvious that $\mathfrak{a} \subset \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}^2$ so $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. To prove the converse inequality, suppose that $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Let $m = x^a y^b z^c$ be a monomial that appears in f. We will argue that m is divisible by one of x^2, xy, xz or yz. Since $x^a y^b z^c \in \mathfrak{m}^2$ we have $a + b + c \geqslant 2$. If $a = 0$ then m is divisible by one of x^2, xy or xz. On the other hand, if $a = 0$ then since $f \in \mathfrak{p}_1$ we have $b > 0$ and since $f \in \mathfrak{p}_2$ we have $c > 0$ so m is divisible by yz. We have proved our assertion. Since $\mathfrak{p}_1\mathfrak{p}_2$ is generated by x^2, xy, xz and yz, we have shown that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 \subseteq \mathfrak{a}$.

The next problem is a strengthening from a fact from last week.

Problem 4. Let $\mathfrak{o}$ be a Dedekind domain with a unique maximal ideal $\mathfrak{p}$. Prove that R is a discrete valuation ring.

Solution. In our earlier exercises we showed that every nonzero ideal of R has been proved to be a product of maximal ideals, so every ideal is of the form $\mathfrak{p}^k$ for some k. Moreover the fractional ideal $\mathfrak{p}^{-1}$ in the field of fractions is not equal to $\mathfrak{o}$. Let ϖ^{-1} be some element of $\mathfrak{p}^{-1} - \mathfrak{o}$. Then $\varpi \in \mathfrak{p}$. If $y \in \mathfrak{o}$ then $(y) = \mathfrak{p}^n$ for some n. Then $(\varpi^{-n}y) = \mathfrak{o}$ so $y = \varpi^n\varepsilon$ where ε is a unit. This shows that every ideal is principal, generated by a power of ϖ, and it follows that $\mathfrak{o}$ is a DVR.

Problem 5. Let A be the quotient of the polynomial ring $k[X, Y]$ by the polynomial $Y^2 - X^3$ and let x, y be the images of X and Y in A. Prove that A is not integrally closed. If $\mathfrak{m}$ is the maximal ideal generated by x and y, determine the dimension of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ for all $n \geqslant 1$.

Solution. The element $t = y/x$ in the field of fractions satisfies the equation $t^2 = x$, so it is integral over A, but it is not in A. The ideal $\mathfrak{m}^n$ with $n > 0$ is generated by $x^a y^b$ with $a + b = n$. Using the relation $y^2 = x^3$, we may dispense with all the generators except two, x^n and yx^{n-1}. These two elements are also a basis of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ which is therefore two-dimensional.

Problem 6. Let λ be a function defined on modules over a ring taking values in $\mathbb{Z} \cup \{\infty\}$ such that if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence then

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

Prove that if $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_4 \longrightarrow 0$ is exact then

$$\lambda(M_1) - \lambda(M_2) + \lambda(M_3) - \lambda(M_4) = 0.$$

Solution. Let M' be the cokernel of the map $M_1 \longrightarrow M_2$. Then we have short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M' \longrightarrow 0, \qquad 0 \longrightarrow M' \longrightarrow M_3 \longrightarrow M_4 \longrightarrow 0,$$

so $\lambda(M_1) - \lambda(M_2) + \lambda(M') = 0$ and $\lambda(M') - \lambda(M_3) + \lambda(M_4) = 0$. Subtracting these identities gives the required result.