Math 210B: Homework 4

All rings are commutative, with 1. Some of the problems make use of the norm map, so review Theorem VI.5.1 of Lang's *Algebra*.

We will say a ring is of *dimension 1* if it is not a field and every nonzero prime ideal is maximal. Recall that a Dedekind domain is an integral domain that is integrally closed (meaning integrally closed in its field of fractions), Noetherian, and of dimension 1.

Problem 1. Let E/F be a finite separable field extension and let $N : E^\times \to F^\times$ be the norm map. Let A be a subring of F that is integrally closed (in F) and let B be the integral closure of A in E. Let $x \in B$. Show that $N(x) \in A$. Moreover show that $x \in B^\times$ if and only if $N(x) \in A^\times$.

Problem 2. Let E/F be a finite separable field extension and let A be a Dedekind domain whose field of fractions is F. Let B be the integral closure of A in E. Prove that B is a Dedekind domain.

Problem 3. Show that a principal ideal domain is a Dedekind domain. Show that $\mathbb{Z}[\sqrt{-5}]$ is Dedekind domain that is not a PID.

Let $\mathfrak{o}$ be a Dedekind domain and let K be its field of fractions. Let $\mathfrak{a}$ be a nonzero $\mathfrak{o}$-submodule of K. If there exists a nonzero $c \in \mathfrak{o}$ such that $c\mathfrak{a} \subset \mathfrak{o}$, then $\mathfrak{a}$ is called a *fractional ideal*. Thus a nonzero ideal of $\mathfrak{o}$ is a fractional ideal.

The next exercise continues (from Homework 3) Exercise VII.7 in Lang's *Algebra*.

Problem 4. Let $\mathfrak{o}$ be a Dedekind domain and let K be its field of fractions. Let $\mathfrak{p}$ be a nonzero maximal ideal of $\mathfrak{o}$. Define

$$\mathfrak{p}^{-1} = \{x \in K | x\mathfrak{p} \subseteq \mathfrak{o}\}.$$

(a) Show that $\mathfrak{p}^{-1}$ is a fractional ideal containing $\mathfrak{o}$ and that $\mathfrak{p}^{-1} \neq \mathfrak{o}$.

(b) Continuing from the preceding problem, show that $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{o}$.

(c) Show that every non-zero ideal is invertible by a fractional ideal. Deduce that the fractional ideals of $\mathfrak{o}$ form a group under multiplication.

Hints: For (a) try to use Problem 6 from Homework 3. If $0 \neq x \in \mathfrak{p}$ show that you can find primes $\mathfrak{p}_i$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq x\mathfrak{o} \subseteq \mathfrak{p}$. Show that one of the $\mathfrak{p}_i = \mathfrak{p}$. Then what? For (b), use the assumption that $\mathfrak{o}$ is integrally closed to rule out the possibility that $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}$.

Problem 5. Let R be a principal ideal domain with a unique nonzero prime ideal. Prove that R is a valuation ring of its field of fractions F.

We will call a principal ideal domain with a unique nonzero prime ideal a *discrete valuation ring* (DVR).