Math 210B: Homework 3

Recall that if $\mathfrak{a}$ is an ideal of A the *radical* of $\mathfrak{a}$ is

$$r(\mathfrak{a}) = \{f \in A | f^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

This is an ideal containing $\mathfrak{a}$, and it is easy to see that $r(r(\mathfrak{a})) = r(\mathfrak{a})$. The ideal $\mathfrak{a}$ is called *radical* if $\mathfrak{a} = r(\mathfrak{a})$.

Let k be a field, which we assume to be algebraically closed. Let $\mathbb{A}^n(k)$ be *affine n-space* k^n. If S is a subset of the polynomial ring $k[X] = k[X_1, \cdots, X_n]$ then

$$V(S) = \{a = (a_1, \cdots, a_n) \in \mathbb{A}^n(k) | f(a) = 0 \text{ for all } f \in S\}.$$

If $\mathfrak{a}$ is the ideal generated by S and $r(\mathfrak{a})$ is the radical of $\mathfrak{a}$ then $V(S) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Also if X is a subset of $\mathbb{A}^n(k)$ let

$$I(X) = \{f \in k[X] | f(a) = 0 \text{ for all } a \in \mathfrak{a}\}.$$

It is an ideal. By the Nullstellensatz $I(V(\mathfrak{a})) = r(\mathfrak{a})$.

The sets $V(S)$ are called *algebraic sets* and last week you proved they form the closed sets in a topology, called the *Zariski topology* on $\mathbb{A}^n(k)$. The essential content of the Nullstellensatz is that $\mathfrak{a} \mapsto V(\mathfrak{a})$ is a bijection between radical ideals and Zariski closed sets.

A closed (i.e. algebraic) subset X of $\mathbb{A}^n(k)$ is called *reducible* if there exist proper closed subsets Y, Z of X such that $X = Y \cup Z$. If it is not reducible it is called *irreducible*. I will call an irreducible closed set a *variety* though this terminology is not universally adopted: often a variety is not required to be irreducible.

Problem 1. Prove that $V(\mathfrak{a})$ is irreducible if and only if $r(\mathfrak{a})$ is prime.

Problem 2. (a) Let $A \subset F$ where F is a field, and let B be the integral closure of A in F. Let $S \subseteq A$ be a multiplicative set. Show that $S^{-1}B$ is the integral closure of $S^{-1}A$ in F.

(b) Let A be an integral domain. We recall that we say A is *integrally closed* if it is integrally closed in its field of fractions F. Show that A is integrally closed if and only if $A_{\mathfrak{m}}$ is integrally closed for every maximal ideal $\mathfrak{m}$ of A.

Hint for (b): Let C be the integral closure of A in F. Let $x \in C$. Suppose that $x \notin A$. Let $\mathfrak{a} = \{f \in A | fx \in A\}$. Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$. Then ...

We recall the *Extension Theorem for valuations* that was proved in Week 2. If F is a field, a *valuation ring of* F is a subring R such that if $x \in F$ then either $x \in R$ or $x^{-1} \in R$. A valuation ring is R a local ring. Its maximal ideal $\mathfrak{p}$ may be characterized as $\{x \in R | x^{-1} \notin R\}$ since in a local ring, the nonunits comprise the unique maximal ideal.

Theorem 1. *Let F be a field and A a subring of F. Let $\phi : A \longrightarrow \Omega$ be a homomorphism of A into an algebraically closed field Ω. Then ϕ may be extended to a homomorphism $\Phi : R \longrightarrow \Omega$ where R is a valuation ring of F.*

This is Corollary 3.3 in Section VII.3 of Lang's *Algebra*. It was also proved in class and used to prove the Nullstellensatz. Use it in the next Exercise.

Problem 3. Let F be a field and A a subring of F. Show that the integral closure of A in F is the intersection of all valuation rings of F containing A.

Hint: To prove that if $x \in F$ is not integral over A then there is a valuation ring R of F such that $x \notin R$, show that $x^{-1}A[x^{-1}]$ is contained in a maximal ideal of $A[x^{-1}]$, then find a way to use the Extension Theorem.

Problem 4. Let F be a field and let R be the polynomial ring $F[X, Y]$ in two variables. Give examples of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that the local ring $R_{\mathfrak{p}}$ is a valuation ring but the local ring $R_{\mathfrak{q}}$ is not.

Problem 5. Let A be an integrally closed integral domain. This means that it is integrally closed in its field of fractions K. Let L/K be a finite separable extension and let B be the integral closure of A in L. Suppose that A is Noetherian. Prove that B is a finitely-generated A-module and deduce that B is also Noetherian.

Hint: This is Problem 3 in the exercises to Chapter VII of Lang (page 353). See Lang for the hint.

A commutative ring R is called a *Dedekind domain* if it is integrally closed in its field of fractions, Noetherian and every nonzero prime ideal is maximal. For example a principal ideal domain is a Dedekind domain. The class of Dedekind domains is important because many important rings are Dedekind domains. For example the integral closure of $\mathbb{Z}$ in a finite extension E of $\mathbb{Q}$ is called the *ring of algebraic integers* in E and it is a Dedekind domain. The affine algebra (coordinate ring) of a nonsingular affine curve is a Dedekind domain.

The next exercise is part of Exercise VII.7 in Lang's *Algebra.* We'll return to this Exercise in Homework 4.

Problem 6. Let $\mathfrak{o}$ be a Dedekind domain. Let K be its field of fractions. Given a nonzero ideal $\mathfrak{a}$ of $\mathfrak{o}$ prove that there exists a product of maximal ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{a}$.