Math 210B: Homework 2

All rings are commutative with 1. Recall that if $A \subseteq B$ are rings and $\mathfrak{p}, \mathfrak{P}$ are prime ideals of A, B respectively we say $\mathfrak{P}$ *lies above* $\mathfrak{p}$ if $\mathfrak{P} \cap A = \mathfrak{p}$.

Problem 1. Let p be a prime, so $(p) = p\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. Determine the primes $\mathfrak{P}$ of the Gaussian integers $\mathbb{Z}[i]$ above (p). Thus determine the number of different $\mathfrak{P}$ and describe $\mathbb{Z}[i]/\mathfrak{P}$ for each $\mathfrak{P}$. Your answer should depend on p modulo 4.

Problem 2. Let $A = \mathbb{Q}[x, y]$ be the polynomial ring $\mathbb{Q}[X, Y]$ modulo the ideal generated by the polynomial $Y^2 - X^2(X + 1)$. This polynomial is irreducible since $X^2(X + 1)$ is not a square in $\mathbb{Q}[X]$, so A is an integral domain. If x, y are the images of X and Y then $y^2 = x^3 + x^2$. Let $\mathfrak{p}$ be the ideal generated by x and y. Let $t = y/x$ in the field of fractions of A.

(a) Show that $\mathfrak{p}$ is maximal and that it is the unique prime ideal of A above the ideal (x) of $\mathbb{Q}[x]$.

(b) Consider the rings $\mathbb{Q}[x] \subseteq \mathbb{Q}[x, y] \subseteq \mathbb{Q}[x, t]$, which are all subrings of the field of fractions of A. How many prime ideals of $\mathbb{Q}[x, t]$ lie above $\mathfrak{p}$?

(c) Show that the ring A is not integrally closed.

Let K be a field. Define $\mathbb{A}^n(K) = K^n$ to be the affine space. You may assume that K is infinite, so that $f(x) = f(x_1, \cdots, x_n)$ in the polynomial ring $K[X] = K[X_1, \cdots, X_n]$ is zero as a polynomial if and only if it is zero as a function on $\mathbb{A}^n(K)$. When we discuss the Nullstellensatz we will assume K is algebraically closed.

If $S \subseteq K[X]$ define

$$V(S) = \{x \in \mathbb{A}^n(K) | f(x) = 0 \text{ for } f \in S\}.$$

We call $V(S)$ an *algebraic set*. Clearly $V(S) = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal generated by S, and indeed $V(S) = V(r(\mathfrak{a}))$ where

$$r(\mathfrak{a}) = \{f \in K[X] | f^n \in A \text{ for some } n > 0\}$$

is the radical of A.

Problem 3. Prove that $\mathbb{A}^n(K)$ has a topology in which the algebraic sets are the closed sets. (This is the *Zariski topology.*)

The *Jacobson radical* $J(A)$ is the intersection of all maximal ideals of A. We encountered it in Nakayama's Lemma.

Problem 4. Prove that the Jacobson radical equals

$$\left\{a \in A | 1 + ab \in A^{\times} \text{ for all } b \in A\right\}.$$

Problem 5. Let A be an integral domain, and Ω a field containing A. If Ω is integral over A prove that A is a field.

Let $A \subseteq B$ be rings, B integral over A, and let S be a multiplicative subset of A. Then by Proposition VII.1.8, $S^{-1}B$ is integral over $S^{-1}A$. Now let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} = A \cap \mathfrak{P}$. Then there is a homomorphism $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{P}}$ and we might hope that $B_{\mathfrak{P}}$ is integral over $A_{\mathfrak{p}}$. The following problem shows that this may not be true.

Problem 6. Let k be a field of characteristic $\neq 2$. Let $A = k[x^2 - 1]$ and $B = k[x]$. Show that B is integral over A. Let $\mathfrak{P} = (x - 1)B$ and $\mathfrak{p} = A \cap \mathfrak{P}$. Prove that $1/(x + 1) \in B_{\mathfrak{P}}$ is not integral over $A_{\mathfrak{p}}$.