Math 210B: Homework 1

Readings in Lang's *Algebra* for these problems: Sections 7.1 and 8.1; also Section 4.2 for Gauss' Lemma and Section 6.5 for norm and trace.

For the first two problems, let R be an integral domain, F its field of fractions, K/F a larger field. Let $\alpha \in K$. As we recall, α is defined to be integral over R if it satisfies a monic polynomial $f(\alpha) = 0$ where

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in R[x].$$

However in this definition we do not require f to be irreducible. So α is also a root of a monic polynomial

$$g(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_0 \in F[x]$$

that is irreducible in $F[x]$. The first two exercises give conditions for $g \in R[x]$.

Problem 1. Suppose that R is a unique factorization domain. Use Gauss' Lemma to show that $g(x) \in R[x]$.

The next problem requires some Galois theory.

Problem 2. Now suppose that R is an integrally closed integral domain and $\mathrm{char}(F) = 0$. Let $\alpha_1, \cdots, \alpha_m$ be the roots of g in a splitting field $E \supseteq K$. Thus $m = \deg(g)$. (a) Explain why the α_i are distinct and

$$g(x) = (x - \alpha_1) \cdots (x - \alpha_m).$$

(b) Prove that $b_i \in R$, so just as in Problem 1, $g \in R[x]$.

Problem 3. Let D be squarefree and consider $\alpha = a + b\sqrt{D} \in \mathbb{Q}\left(\sqrt{D}\right)$. Describe the norm and trace $\mathrm{tr}(\alpha)$ and $N(\alpha)$. Prove that α is integral over $\mathbb{Z}$ if and only if $\mathrm{tr}(\alpha)$ and $N(\alpha)$ are in $\mathbb{Z}$.

Problem 4. Let p be a prime. Determine the integral closure of $\mathbb{Z}$ in $\mathbb{Q}\left(\sqrt{p}\right)$ and $\mathbb{Q}\left(\sqrt{-p}\right)$. The shape of the answer should depend on p mod 4.

Problem 5. Let $E \supset K \supset F$ fields. Prove that the transcendence degrees are additive:

$$\mathrm{tr}.\deg(E/F) = \mathrm{tr}.\deg(E/K) + \mathrm{tr}.\deg(K/F).$$