The Going Up and Going Down Theorems

The *Going Up and Going Down* theorems describe the behavior of prime ideals in integral extensions. They were proved by Cohen and Seidenberg in 1946. They have geometric meaning in terms of dimension.

All rings are commutative with 1.

The Going Up Theorem

The Going Up Theorem is the easier of the two. Most of it is already proved in Lang, Chapter VII.

Theorem 1 (The Going Up Theorem)**.** *Let $A \subset B$ be commutative rings with B integral over A. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be prime ideals of A and let $\mathfrak{P}_1$ be a prime ideal of B above $\mathfrak{P}_1$. Then B has a prime ideal $\mathfrak{P}_2$ above $\mathfrak{p}_2$ such that $\mathfrak{P}_1 \subset \mathfrak{P}_2$.*

Proof. We apply Proposition VII.1.11 on page 339 to the pair of rings $A/\mathfrak{p}_1 \subseteq B/\mathfrak{P}_1$. Let $\overline{\mathfrak{p}}_2$ be the image of $\mathfrak{p}_2$ in $A/\mathfrak{p}_1$. The Proposition guarantees the existence of a prime ideal $\overline{\mathfrak{P}}_2$ of $B/\mathfrak{P}_1$ above $\overline{\mathfrak{p}}_2$. Pulling this ideal back to B gives $\mathfrak{P}_2$. □

Here is a slight strengthening of an assertion in Lang's Proposition VII.1.10.

Proposition 2. *Let A be an integral domain and B a ring that is integral over A. Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{p}B \cap A = \mathfrak{p}$.*

Proof. Let $A_\mathfrak{p} = S^{-1}A$ and $B_\mathfrak{p} = S^{-1}B$ where $S = A - \mathfrak{p}$. Then $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and $B_\mathfrak{p}$ is integral over $A_\mathfrak{p}$. By Proposition 1.10 of Lang, $\mathfrak{p}B_\mathfrak{p}$ is a proper ideal of $B_\mathfrak{p}$ and $\mathfrak{p}B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. However since $\mathfrak{p}$ is prime, $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$ and so intersecting the identity $\mathfrak{p}B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ gives $\mathfrak{p}B \cap A \subseteq \mathfrak{p}B_\mathfrak{p} \cap A = \mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$. □

Here is a strengthening of Lang's Proposition VII.1.11.

Proposition 3. *Let $A \subseteq B$ be rings with B integral over A. Let $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ be prime ideals of B. If $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.*

Proof. Let $S = A - \mathfrak{p}$ and consider $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ where $A_\mathfrak{p} = S^{-1}A$ and $B_\mathfrak{p} = S^{-1}B$. This is an integral extension. We have $\mathfrak{q}_1 B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{q}_2 B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ which is a maximal ideal. By Proposition VII.1.11 in Lang's *Algebra* it follows that $\mathfrak{q}_1 B_\mathfrak{p}$ is maximal. Since $\mathfrak{q}_1 B_\mathfrak{p} \subseteq \mathfrak{q}_2 B_\mathfrak{p}$ these ideals are equal. Restricting to B, we get $\mathfrak{q}_1 = \mathfrak{q}_2$. □

The Going Down Theorem

Let $A \subseteq B$ be commutative rings and let $\mathfrak{a}$ be an ideal of A. Let $x \in B$. We say that x is *integral over* $\mathfrak{a}$ if it satisfies a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0, \qquad a_i \in \mathfrak{a}. \tag{1}$$

Proposition 4. *Assume that A and B are integral domains with B integral over A. Then*

$$\{x \in B | x \textit{ is integral over } \mathfrak{a}\} = r(B\mathfrak{a}).$$

Proof. First suppose that $x \in B$ is integral over $\mathfrak{a}$. Rearranging the equation (1) as follows:

$$x^n = -(a_{n-1}x^{n-1} + \ldots + a_0)$$

the right-hand side is in $B\mathfrak{a}$ so $x \in r(B\mathfrak{a})$.

Conversely, we show that every element of $r(B\mathfrak{a})$ is integral over $\mathfrak{a}$. It follows from the definition that if x^n is integral over $\mathfrak{a}$ then x is integral over $\mathfrak{a}$ then so is x. Therefore it is sufficient to show that every element of $B\mathfrak{a}$ is integral over $\mathfrak{a}$.

Thus let $x \in B\mathfrak{a}$. We will prove that x is integral over $\mathfrak{a}$. We may assume that $x \neq 0$. We define an ideal $\mathfrak{b}$ of $A[x^{-1}]$, a subring of the field of fractions E of B. Let

$$\mathfrak{b} = \{y \in A[x^{-1}] | yx \in \mathfrak{a}A[x^{-1}]\}.$$

We will show that $\mathfrak{b} = A[x^{-1}]$. If not, $\mathfrak{b}$ is a proper ideal and we may embed it in a maximal ideal $\mathfrak{m}$ of $A[x^{-1}]$. Consider the canonical homomorphism $\phi : A[x^{-1}] \longrightarrow A[x^{-1}]/\mathfrak{m}$. Let Ω be the algebraic closure of $A[x^{-1}]/\mathfrak{m}$. By the Extension Theorem (Corollary VII.3.3 on page 348 of Lang) we may extend ϕ to a homorphism $\Phi : R \longrightarrow \Omega$ where R is a valuation ring of E containing $A[x^{-1}]$.

Note that R contains all of B since B is integral over A, hence over $A[x^{-1}]$ by Homework 3, Problem 3. In particular R contains x but it also contains x^{-1} since $A[x^{-1}] \subseteq R$. Thus x is a unit in R and since $\ker(\Phi)$ consists of the nonunits of R, we see that $\Phi(x) \neq 0$. Now $x^{-1}\mathfrak{a} \subseteq \mathfrak{b}$ by the definition of $\mathfrak{b}$ so $\Phi(x^{-1}\mathfrak{a}) = 0$. Since $\Phi(x) \neq 0$ it follows that $\Phi(\mathfrak{a}) = 0$ and so $\Phi(\mathfrak{a}B) = 0$. Since $x \in \mathfrak{a}B$ and $\Phi(x) \neq 0$, this is a contradiction. This contradiction proves that $\mathfrak{b} = A[x^{-1}]$.

In particular $1 \in \mathfrak{b}$ which, from the definition of $\mathfrak{b}$ means that $x \in A[x^{-1}]$. We may therefore write

$$x = a_0 + a_1x^{-1} + \cdots + a_nx^{-n}, \qquad a_i \in \mathfrak{a}.$$

It follows that

$$x^{n+1} - a_0x^n - \ldots - a_n = 0,$$

so x is integral over $\mathfrak{a}$. □

Proposition 5. *Suppose that A is an integral domain that is integrally closed in its field of fractions F. Let E be a field containing F, and let $x \in E$. Suppose that x is integral over over A. Let*

$$X^n + a_{n-1}X^{n-1} + \ldots + a_0$$

be the monic irreducible polynomial satisfied by x over F. Then the $a_i \in A$. Moreover if $\mathfrak{p}$ is a prime ideal of A and if x is integral over $\mathfrak{p}$ then the $a_i \in \mathfrak{p}$.

Proof. The fact that the $a_i \in A$ is Problem 2 in Homework 2. To prove the last statement, let L be an extension of E containing all the conjugates $\sigma_i(x)$, as σ runs through the embeddings of E into $\overline{E}$ over F. Let B be the integral closure of A in L.

$$X^n + a_{n-1}X^{n-1} + \ldots + a_0 = \prod (X - \sigma_i(x)).$$

Now all the $\sigma_i(x)$ are integral over $\mathfrak{p}$ so by our previous Proposition they lie in $r(\mathfrak{p}B)$. Hence the coefficients a_i lie in

$$r(\mathfrak{p}B) \cap A \subseteq r(\mathfrak{p}B \cap A) = r(\mathfrak{p}) = \mathfrak{p}$$

where we have used Proposition 2. Hence $a_i \in \mathfrak{p}$. □

Lemma 6. *Let $A \subset B$ be rings and let $\mathfrak{p}$ be a prime ideal of A. Assume that $\mathfrak{p}B \cap A = \mathfrak{p}$. Then there exists a prime ideal $\mathfrak{P}$ of B above $\mathfrak{p}$.*

Note that we are *not* assuming that B is integral over A here.

Proof. Let $S = A \backslash \mathfrak{p}$ and let $B_\mathfrak{p} = S^{-1}B$. Since $\mathfrak{p} \cap S = \varnothing$ the ideal $\mathfrak{p}B_\mathfrak{p}$ is proper, so it is contained in a maximal ideal $\mathfrak{m}$ of $B_\mathfrak{p}$. Let $\mathfrak{P}$ be the preimage of $\mathfrak{m}$ under the canonical homomorphism $B \to B_\mathfrak{p}$. Then $A \cap \mathfrak{P}$ is the preimage of $\mathfrak{m}$ under the canonical homomorphism $A \to B_\mathfrak{p}$, which is the composition $A \to A_\mathfrak{p} \to B_\mathfrak{p}$. Thus $A \cap \mathfrak{P} = \mathfrak{p}A_\mathfrak{p} = \mathfrak{p}$. □

Theorem 7 (The Going Down Theorem)**.** *Let $A \subseteq B$ be integral domains with A integrally closed in its field of fractions and B integral over A. Suppose that $\mathfrak{p}_1 \subset \mathfrak{p}_2$ are prime ideals of A and $\mathfrak{q}_2$ a prime ideal of B over $\mathfrak{p}_2$. Then there exists a prime ideal $\mathfrak{q}_1$ of B over $\mathfrak{p}_1$ such that $\mathfrak{q}_1 \subset \mathfrak{q}_2$.*

Proof. Let F and E be the fields of fractions of A and B respectively.

We begin by showing that $\mathfrak{p}_1 B_{\mathfrak{q}_2} \cap A = \mathfrak{p}_1$. The $\supseteq$ inclusion is obvious, so let $x \in \mathfrak{p}_1 B_{\mathfrak{q}_2} \cap A$. We may write $x = y/s$ where $x \in \mathfrak{p}_1 B$ and $s \in B \backslash \mathfrak{q}_2$. Let

$$X^r + u_1 X^{r-1} + \ldots + u_r$$

be the minimal polynomial of y over F. The coefficients u_i are in $\mathfrak{p}_1$ by Proposition 5. Writing $s = y/x$, the minimal polynomial for s is

$$X^r + v_i X^{r-1} + \ldots + v_r, \qquad v_i = u_i/x^i.$$

The coefficients $v_i \in A$ since $s \in B$ is integral over A and A is integrally closed.

Now we will argue that $x \in \mathfrak{p}_1$. If not, then $u_i = v_i x^i$ and since $v_i \in B, x^i \notin \mathfrak{p}_1$ but $u_i \in \mathfrak{p}_1$ we must have $v_i \in \mathfrak{p}_1$. Now

$$s^r = -(v_1 s + \cdots + v_r) \in \mathfrak{p}_1 B \subseteq \mathfrak{p}_2 B \subseteq \mathfrak{q}_2$$

and since $\mathfrak{q}_2$ is prime, $s \in \mathfrak{q}_2$, which is a contradiction. This completes the proof that $\mathfrak{p}_1 B_{\mathfrak{q}_2} \cap A = \mathfrak{p}_1$.

Now by the Lemma there exists a prime $\mathfrak{q}$ of $B_{\mathfrak{q}_2}$ above $\mathfrak{p}_1$. We take $\mathfrak{q}_1 = B \cap \mathfrak{q}$ and we are done. $\square$