Dimension I

Dimension and Combinatorial Dimension

We will discuss some aspects of the theory of dimension in algebraic geometry. This theory was created mainly by Wolfgang Krull in the 20th century.

Let X be an affine variety over an algebraically closed field k, with coordinate ring $A = \mathcal{O}(X)$. Thus X is required to be irreducible and so A is an integral domain that is reduced and finitely generated over k. We define the *dimension* $\dim(X)$ to be the transcendence degree of the field F of fractions of A over k.

Alternatively we may consider chains of closed, nonempty, irreducible subsets of X such as:

$$X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_d.$$

The chain is *saturated* if no further closed irreducible subsets may be inserted to make a longer chain. Clearly the chain is saturated, X_0 must consist of a single point, and X_d must equal X. The supremum of the lengths d of such chains is called the *combinatorial dimension*. Our goal is to show that $\dim(X)$ equals the combinatorial dimension.

If A is a commutative ring, the *Krull dimension* of A is the supremum of the maximal lengths d of chains of prime ideals:

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_d$$

If the chain is saturated then clearly $\mathfrak{p}_d$ is maximal and furthermore if A is an integral domain then $\mathfrak{p}_0 = 0$.

We recall that there is an inclusion reversing bijection between prime ideals of A and closed subsets of X. Therefore the combinatorial dimension of X is the Krull dimension of $A = \mathcal{O}(X)$. The key step is Proposition 2 below, whose proof showcases typical applications of both the Noether normalization theorem and the going-down theorem.

Lemma 1. *Suppose that R is a unique factorization domain and $\mathfrak{p}$ a minimal nonzero prime ideal of R. Then $\mathfrak{p} = (f)$ is a principal ideal generated by an irreducible element f of R.*

Proof. Since $\mathfrak{p} \neq 0$ it contains principal ideals (f), and since it is Noetherian let (f) be a maximal such principal ideal. We claim that f is irreducible. If not, write $f = f_1 f_2$ where f_1 and f_2 are nonunits. Then $f_1 \in \mathfrak{p}$ or $f_2 \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. Thus $(f) \subsetneqq (f_1) \subseteq \mathfrak{p}$ contradicting the maximality of (f). This proves that f is irreducible. Then (f) is a nonzero prime ideal contained in $\mathfrak{p}$ and by the assumed maximality of $\mathfrak{p}$ we have $\mathfrak{p} = (f)$. □

Proposition 2. *Suppose that X is an irreducible affine variety. Then X has maximal proper closed irreducible subsets. Suppose that Y is such a maximal proper closed irreducible subset of X. Then* $\dim(Y) = \dim(X) - 1$.

Proof. Since $B = \mathcal{O}(X)$ is finitely generated let $A = k[x_1, \cdots, x_n]$. We order the generators so that $x_1, \cdots, x_d$ are algebraically independent over k and $x_{d+1}, \cdots, x_d$ are algebraic over $k(x_1, \cdots, x_d)$. By definition $d = \dim(X)$. By the Noether normalization theorem (Theorem VIII.2.1 in Lang's *Algebra*) we may choose the generators x_i so that B is integral over $A = k[x_1, \cdots, x_d]$, which is a polynomial ring. Thus it is A is a unique factorization domain and if f is any irreducible element then $\mathfrak{p} = (f)$ is a minimal nonzero prime ideal. By the Going-Up theorem there is a prime ideal $\mathfrak{P}$ of B above $\mathfrak{p}$. Thus the ideals $\mathfrak{P} \supsetneqq (0)$ are above the primes $\mathfrak{p} \supsetneqq (0)$ of A. We claim that $\mathfrak{P}$ is a minimal nonzero prime ideal. Indeed, if $\mathfrak{P} \supsetneqq \mathfrak{Q} \supsetneqq (0)$ then by Proposition 3 of our note on the Going Up and Going Down Theorems, $\mathfrak{p} \supsetneqq \mathfrak{Q} \cap A \supsetneqq (0)$ contradicting the minimality of $\mathfrak{p}$.

Let Y be the subvariety inside X corresponding to $\mathfrak{P}$. Since its ideal is a minimal nonzero prime ideal, Y is a maximal proper irreducible closed subset of X.

We have proved that X has maximal proper closed irreducible subset. But we actually need to know that every maximal proper closed subset is arises this way. Thus if Y is given, its prime ideal $\mathfrak{P}$ is a minimal nonzero prime of B, so by Proposition 3 of our note on the Going Up and Going Down Theorems, $\mathfrak{p} = \mathfrak{P} \cap A$ is a minimal nonzero prime ideal of the unique factorization domain A, that is $\mathfrak{p} = (f)$ for some irreducible $f \in A$.

We order $x_1, \cdots, x_d$ so that f involves x_d nontrivially. Let $\bar{x}_i$ denote the images of x_i in $A/(\mathfrak{P} \cap A) = A/(f)$. We note that $\bar{x}_1, \cdots, \bar{x}_{d-1}$ are algebraically independent since the kernel (f) of the projection $A \longrightarrow A/(f)$ involves x_d nontrivially. On the other hand, f gives a relation of algebraic dependence of $\bar{x}_d$ over $\bar{x}_1, \cdots, \bar{x}_{d-1}$. Therefore $\bar{x}_1, \cdots, \bar{x}_{d-1}$ are a transcendency basis of the field of fractions of $A/(f)$. Furthermore $B/\mathfrak{p}$ is integral over $A/(f)$ since B is integral over A. Thus $\bar{x}_1, \cdots, \bar{x}_{d-1}$ are a transcendency basis of the field of fractions of $B/\mathfrak{p} = \mathcal{O}(Y)$, proving that $\dim(Y) = d - 1$. □

Theorem 3. *Let X be an affine variety. Then* $\dim(X)$ *equals the combinatorial definition of X. Indeed every saturated chain of closed irreducible subsets has length exactly d.*

Proof. Indeed, let $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_d = X$ be a saturated chain of closed irreducible subsets. Then $\dim(X_{d-1}) = \dim(X_d) - 1$ by Proposition 2. Note that $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{d-1}$ is a saturated chain for X_{d-1}, so by induction $\dim(X_{d-1}) = d-1$. Therefore $\dim(X) = d$. □

Nonsingular Plane Curves

An (affine) algebraic variety of dimension one is called a *curve*. From Theorem 3, an affine variety X is of dimension one if and only if $\mathcal{O}(X)$ is of Krull dimension one, that is, if every nonzero prime ideal is maximal. If furthermore $\mathcal{O}(X)$ is integrally closed, then it is a Dedekind domain. In this section we will study *plane curves*, that is, curves embedded in $\mathbb{A}^2$.

We may define a *discrete valuation ring* to be a principal ideal domain R with a unique maximal ideal $\mathfrak{p}$. Let $\varpi \in \mathfrak{p}$ be a generator. Then every nonzero element of R may be uniquely written as $\varpi^k\varepsilon$ where $\varepsilon \in R^\times$ and $k \geqslant 0$. If F is the field, we may again write every element as $\varpi^k\varepsilon$ where $\varepsilon \in R^\times$ but now k is allowed to be negative. The map $\mathrm{ord}_\mathfrak{p} : R \longrightarrow \mathbb{Z}$ that maps $\varpi^k\varepsilon$ to k is called the *valuation*. The parameter ϖ is sometimes called a *local parameter* or *uniformizer*.

Proposition 4. **??** *Let R be a Noetherian local domain with maximal ideal $\mathfrak{m}$. Suppose that $\mathfrak{m}$ is principal: $\mathfrak{m} = \varpi R$. Then R is a discrete valuation ring.*

Proof. Let $\mathfrak{a}$ be any nonzero ideal of R. We will argue that $\mathfrak{a} = \mathfrak{m}^k$ for some k. If $\mathfrak{a} = R$ we may take $k = 0$, so assume that $\mathfrak{a}$ is proper. Then $\mathfrak{a} \subseteq \mathfrak{m}$ and so $\varpi^{-1}\mathfrak{a} \subseteq \varpi^{-1}\mathfrak{m} = R$. We claim that $\varpi^{-1}\mathfrak{a} \neq \mathfrak{a}$. Indeed if $\varpi^{-1}\mathfrak{a} = \mathfrak{a}$ then $\mathfrak{a} = \varpi\mathfrak{a} = \mathfrak{m}\mathfrak{a}$ and since $\mathfrak{a}$ is finitely generated $\mathfrak{a} = 0$ by Nakayama's Lemma. This is a contradiction. Thus $\varpi^{-1}\mathfrak{a}$ is a strictly larger ideal than $\mathfrak{a}$ and by induction (since R is Noetherian) $\varpi^{-1}\mathfrak{a} = \mathfrak{m}^{k-1}$ for some integer k.

Since $\mathfrak{a} = \mathfrak{m}^k = (\varpi^k)$ we have proved that every ideal is principal so R is a PID. □

Let f be an irreducible polynomial in the polynomial ring $k[X, Y]$. Let

$$C = \{(a, b) \in \mathbb{A}^2 | f(a, b) = 0\}.$$

This is a plane curve with coordinate ring $A = \mathcal{O}(C) = k[X, Y]/(f)$. If (a, b) is a point of C we say (a, b) is *nonsingular* if either

$$\frac{\partial f}{\partial X}(a, b) \neq 0 \qquad \text{or} \qquad \frac{\partial f}{\partial Y}(a, b) \neq 0.$$

This is a provisional definition, since later we will see that the property of being nonsingular is intrinsic, and does not depend on the embedding of C into affine space.

Proposition 5. *The plane curve C has only finitely many singular points.*

Proof. It is not possible for $\partial f/\partial X$ and $\partial f/\partial Y$ to both be identically zero unless the characteristic of the ground field k is a prime p, and every term in f is a power of p, that is

$$f(X, Y) = \sum_{m,n} a_{m,n}(X^m Y^n)^p.$$

Since k is algebraically closed, we may find $b_{m,n}$ such that $b^p_{m,n} = a_{m,n}$, and then $f = f_1^p$ where

$$f_1(X, Y) = \sum_{m,n} b_{m,n}(X^m Y^n).$$

This is a contradiction since f is irreducible.

Now by symmetry we may assume that $\partial f/\partial Y$ is not identically zero. Since f is irreducible, the polynomials f and $\partial f/\partial Y$ are coprime in $k[X, Y]$, so they can vanish simultaneously at only a finite number of points in $\mathbb{A}^2$. The singular points must be among these. □

We will denote by x and y the images of X, Y in A so that $f(x, y) = 0$.

Proposition 6. *Suppose that* $\frac{\partial f}{\partial Y}(a, b) \neq 0$. *Let* $\mathfrak{m}$ *be the maximal ideal of* A *consisting of functions vanishing at* (a, b). *Then the local ring* $A_{\mathfrak{m}}$ *is a discrete valuation ring, and the maximal ideal* $\mathfrak{m}A_{\mathfrak{m}}$ *of* $A_{\mathfrak{m}}$ *is the principal ideal generated by* x.

Proof. Replacing $f(X, Y)$ by $f(X - a, Y - b)$ we may assume that $a = b = 0$.

It is enough to show that if $g \in A$ vanishes at (a, b) then g is a multiple of x in $A_{\mathfrak{m}}$. Indeed, if we know this then $\mathfrak{m} \subseteq xA_{\mathfrak{m}}$ and so $\mathfrak{m}A_{\mathfrak{m}} \subseteq xA_{\mathfrak{m}} \subseteq \mathfrak{m}A_{\mathfrak{m}}$ proving that $\mathfrak{m}A_{\mathfrak{m}}$ is principal, hence a discrete valuation ring by Proposition **??**.

Since $g(0, 0) = 0$ the one-variable polynomial $g(0, Y)$ vanishes at $Y = 0$. Therefore may write $g(0, Y) = Yg_1(Y)$ where $Y \in k[Y]$. Similarly $f(0, 0) = 0$ so $f(0, Y) = Yf_1(Y)$. Now consider the polynomial

$$f_1(Y)g(X, Y) - g_1(Y)f(X, Y).$$

This vanishes when $X = 0$ and so it is a multiple of X and we may write

$$f_1(Y)g(X, Y) - g_1(Y)f(X, Y) = Xh(X, Y)$$

in $k[X, Y]$. On substituting x, y for X, Y we obtain $f_1(y)g(x, y) = xh(x, y)$.

Since $\frac{\partial f}{\partial Y}(0, 0) \neq 0$ we have $f_1(0) \neq 0$. Thus $f_1(y)$ is invertible in $A_{\mathfrak{m}}$ and so $g(x, y)$ is a multiple of x. □

Theorem 7. *Let* C *be the plane curve defined by the equation* $f(X, Y) = 0$. *The local ring of* C *at any nonsingular point is a discrete valuation ring.*

Proof. If $\partial f/\partial Y \neq 0$ at (a, b), this follows from Proposition 6. If $\partial f/\partial X \neq 0$ we interchange the roles of X and Y. □

Suppose on the other hand that (a, b) is a singular point. Changing coordinates so that $(a, b) = (0, 0)$ the condition that $f(0, 0) = \partial f/\partial X(0, 0) = \partial f/\partial Y(0, 0) = 0$ means that the Taylor expansion of the polynomial f at $(0, 0)$ looks like this:

$$f(X, Y) = aX^2 + bXY + cY^2 + \text{higher order terms}.$$

Now the local ring is definitely not a discrete valuation ring, for the maximal ideal $\mathfrak{m}$ is not principal; it is generated by x and y and neither can be dispensed with.

Conclusion: For a nonsingular point on an affine curve in $\mathbb{A}^2$, the local ring is a discrete valuation ring. The maximal ideal is principal, generated by a singular element. For singular points, the maximal ideal is not principal and requires at least two generators.

Nonsingular Points

Let X be an affine variety of dimension d, and let $a \in X$. Let $\mathfrak{m} = \mathfrak{m}_a$ be the maximal ideal of all $f \in A = \mathcal{O}(X)$ such that $f(a) = 0$. Then $A/\mathfrak{m}$ is a field, and indeed $A/\mathfrak{m}$ is isomorphic to k, since $\mathfrak{m}$ is the kernel of the homomorphism $f \mapsto f(a)$ from A to k.

Note that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over $A/\mathfrak{m} \cong k$. If $\mathfrak{M} = \mathfrak{m}A_\mathfrak{m}$ is the maximal ideal of the local ring it may be identified with $\mathfrak{M}/\mathfrak{M}^2$.

Proposition 8. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $x_1, \cdots, x_d \in \mathfrak{M}$ be such that their images in $\mathfrak{M}/\mathfrak{M}^2$ span this vector space over the field $A/\mathfrak{M}$. Then the x_i generate the ideal $\mathfrak{M}$.*

Proof. Let $\mathfrak{N}$ be the ideal generated by the x_i. Every element of $\mathfrak{M}$ is congruent modulo $\mathfrak{M}^2$ to an element of $\mathfrak{N}$, so $\mathfrak{M}^2 + \mathfrak{N} = \mathfrak{M}$. This implies that $\mathfrak{M}(\mathfrak{M}/\mathfrak{N}) = \mathfrak{M}/\mathfrak{N}$. By Nakayama's Lemma $\mathfrak{M}/\mathfrak{N} = 0$. Therefore $\mathfrak{N} = \mathfrak{M}$. □

Applying this to the case at hand, if x_i are a basis of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over k, then their images generate the maximal ideal $\mathfrak{M}$ of $A_\mathfrak{M}$.

The *Zariski tangent space* is the dual space $(\mathfrak{m}/\mathfrak{m}^2)^*$ as a k-vector space. To see why this vector space may be identified with the tangent space, imagine that we embed the variety in affine n space, and let x_i be the coordinate function. The tangent vector $\partial/\partial x_i$ may be applied to a function $f \in A$ near a point a. Thus define

$$D_i(f) = \frac{\partial f}{\partial x_i}(a).$$

This map $D_i : A \longrightarrow k$ is a *derivation* satisfying $D_i(fg) = f(a)D_i(g) + g(a)D_i(f)$. Now let $\mathfrak{m}$ be the maximal ideal of functions vanishing at a. Then clearly $D_i(\mathfrak{m}^2) = 0$. So D_i induces a liner functional on $\mathfrak{m}/\mathfrak{m}^2$, that is, an element of $(\mathfrak{m}/\mathfrak{m}^2)^*$.

We have seen from our examination of curves that the dimension of $\mathfrak{m}/\mathfrak{m}^2$ can be equal to the dimension of X and it can also be greater. We have not yet proved that $\mathfrak{m}/\mathfrak{m}^2 \geq \dim(X)$ but at least we have given examples to show that $\mathfrak{m}/\mathfrak{m}^2$ may be either equal to or greater than $\dim(X)$.

Definition 9. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. If the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over the field $A/\mathfrak{m}$ equals the Krull dimension of A then A is called a **regular local ring**.

We may now define a point of a variety to be *smooth* if its local ring is a regular local ring. For plane curves, this agrees with the definition we gave previously. At least, we proved in Theorem 7 that the local ring at a nonsingular point is a DVR and therefore is integrally closed.