Homework 7 Solutions

November 30, 2016

Problems 1-5 in Chapter 20 of Lang's *Algebra*.

1. *Prove that the standard complex is actually a complex, and is exact, so that it gives a resolution of* $\mathbb{Z}$.

Solution. To show that $d^2 = 0$, apply $d_{i-1}d_i$ to $(x_0, \cdots, x_i)$. The terms are of the form $(x_0, \cdots, \widehat{x_j}, \cdots, \widehat{x_k}, \cdots, x_i)$ because each application of d eliminates one term. Each term occurs twice (since we may eliminate the x_j and x_k in either order) and with opposite signs, so the cancel. This the standard complex is a complex, but we still have to show that $\ker(d_i) \subseteq \mathrm{im}(d_{i+1})$.

The standard resolution depends on the choice of a nonempty set S. Pick an element $z \in S$. Define a map $h : E^i \longrightarrow E^{i+1}$ that sends $(x_0, \cdots, x_i)$ to $(z, x_0, \cdots, x_i)$. Let us check that

$$dh + hd = 1_{E_i}. \tag{1}$$

Indeed, applying hd to $(x_0, \cdots, x_i)$ gives

$$(\hat{z}, x_0, \cdots, x_i) + \sum_{j=0}^{i} (-1)^{j+1}(z, x_0, \cdots, \widehat{x_j}, \cdots, x_i),$$

where the hat denotes the omission of a term, so that the first term is really $(x_0, \cdots, x_i)$. Applying hd to $(x_0, \cdots, x_i)$ gives

$$\sum_{j=0}^{i} (-1)^{j}(z, x_0, \cdots, \widehat{x_j}, \cdots, x_i).$$

Adding these two the terms cancel in pairs except one, proving (1).

Now we may prove that the complex is exact. Let $\xi \in \ker(d_i)$. Then by (1), $\xi = d_{i+1}h_i(\xi) + h_{i-1}d_i(\xi) = d_{i+1}h_i(\xi)$, so $\xi \in \mathrm{im}(d_{i+1})$.

2. *Let G be a group. Use G as the set S in the standard complex. Define an action of G on the standard complex by leting $x(x_0, \cdots, x_i) = (xx_0, \cdots, xx_i)$. Prove that each E^i is a free module over the group ring $\mathbb{Z}[G]$. Thus if we let $R = \mathbb{Z}[G]$ be the group ring and consider the category $\mathrm{Mod}(G)$ of G-modules, then the standard complex gives a free resolution of $\mathbb{Z}$ in this category.*

Solution. We say that a G-module A is *trivial* if $gx = g$ for all $g \in G$ and $x \in A$. Here $\mathbb{Z}$ is a trivial G-module.

The abelian group E^i is the free $\mathbb{Z}$-module on $\mathbb{Z}[G]^{i+1}$. The elements $(x_0, \cdots, x_i)$ of $\mathbb{Z}[G]^{i+1}$ thus consist of a basis of E_i as a $\mathbb{Z}$-module. To see that E^i is free as a G-module, let $x_1, \cdots, x_i \in G$ and define $E^i(x_1, \cdots, x_i)$ to be the $\mathbb{Z}[G]$-submodule generated by $(1, x_1, \cdots, x_i)$. Then every basis element $(x_0, \cdots, x_i)$ lies in a unique such module, namely $E^i(x_0^{-1}x_1, \cdots, x_0^{-1}x_i)$. Therefore

$$E^i = \bigoplus_{(x_1, \cdots, x_i) \in \mathbb{Z}[G]^i} E_i(x_1, \cdots, x_i) \tag{2}$$

and so E_i is a free $\mathbb{Z}[G]$-module with basis consisting of the elements $(1, x_1, \cdots, x_i)$.

Now consider the standard complex

$$\ldots \longrightarrow E^1 \longrightarrow E^0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

It follows from the definitions of the maps $d_i : E^i \longrightarrow E^{i-1}$ and the augmentation map $\varepsilon : E^0 = \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ that these are all $\mathbb{Z}[G]$-module homomorphisms, so this is a free resolution of the trivial module $\mathbb{Z}$.

Problem 1 shows that the complex is exact, even though its proof makes use of a map h that is not a $\mathbb{Z}[G]$-module homomorphism!

3. *The standard complex E was written in homogeneous form, so the boundary maps have a certain symmetry. There is another complex that exhibits useful features as follows. Let F^i be the free $\mathbb{Z}[G]$-module having for basis i-tuples (rather than $(i+1)$-tuples) $(x_1, \cdots, x_i)$ For $i = 0$ we take $F_0 = \mathbb{Z}[G]$ itself. Define the boundary operator*

$$d(x_1, \cdots, x_i) = x_1(x_2, \cdots, x_i) + \sum_{j=1}^{i-1}(-1)^j(x_1, x_2, \cdots, x_jx_{j+1}, \cdots, x_i)$$
$$+(-1)^{i+1}(x_1, x_2, \cdots, x_{i-1}).$$

(The last term is misprinted in some copies of Lang.) Show that $E \cong F$ as complexes of G-modules via the map $F^i \longrightarrow E^i$ in which

$$(x_1, \cdots, x_i) \mapsto (1, x_1, x_1x_2, \ldots, x_1 \cdots x_i). \tag{3}$$

Proof. Let $\theta_i : F^i \longrightarrow E^i$ be the map (3). It follows from (2) that θ_i maps the given basis of F^i onto a basis of E^i. Thus the map (3) is an isomorphism of $\mathbb{Z}[G]$-modules. We thus need to compute $\theta_i^{-1} d_i \theta_i$ and check that it has the advertized formula. Consider what happens when we omit the j-th component term from $(1, x_1, x_1 x_2, \ldots, x_1 \cdots x_i) \in E^i$. If $j = 0$ we get

$$(x_1, x_1 x_2, \ldots, x_1 \cdots x_i) = x_i(1, x_2, \cdots, x_2 \cdots x_i) = x_i \theta_{i-1}(x_2, \cdots, x_i).$$

On the other hand if $0 < j < i$, we get

$$(1, x_1, x_1 x_2, \ldots, x_1 \cdots x_{j-1}, x_1 \cdots x_{j+1}, \ldots) = \theta_{i-1}(x_1, x_2, \cdots, x_j x_{j+1}, \cdots, x_i).$$

Finally if $j = i$ we get $\theta_{i-1}(x_1, x_2, \cdots, x_i)$. From these calculations, we see that $\theta_i^{-1} d_i \theta_i$ is given by the advertized formula. □

4. *If A is a G-module, let A^G be the submodule consisting of all elements $v \in A$ such that $xv = v$ for all $x \in G$. Thus A^G (sometimes called the module of invariants) has trivial G-action.*

(a) Show that if $H^q(G, A)$ denotes the q-th homology of the complex $\mathrm{Hom}_G(E, A)$*, then $H^0(G, A) = A^G$. Thus the left derived functors of $A \mapsto A^G$ are the homology groups of the complex* $\mathrm{Hom}_G(E, A)$*, or for that matter, of the complex* $\mathrm{Hom}_G(F, A)$*, where F is as in Exercise 3.*

(b) Show that the group of 1-cycles $Z^1(G, A)$ consists of those functions $f : G \longrightarrow A$ satisfying

$$f(xy) = f(x) + xf(y), f(xy) = f(x) + xf(y), \qquad x, y \in G. \tag{4}$$

Show that the subgroup of coboundaries $B^1(G, A)$ consists of those functions f for which theere exists an element $a \in A$ such that $f(x) = xa - a$. The factor group is then $H^1(G, A)$.

(c) Show that the group of 2-cocycles $Z^2(G, A)$ consists of those functions $f : G \longrightarrow A$ satisfying

$$f(xy) = f(x) + xf(y), \qquad x, y \in G.$$

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0. \tag{5}$$

Such 2-cocycles are also called *factor sets*, and they can be used to describe isomorphism classes of group extensions, as in Exercise 5.

Solution: In (b) and (c), Lang is using the resolution F. (Other resolutions might be useful and give different descriptions.) For (a), note that $\mathrm{Hom}_G(\mathbb{Z}, A) \cong A^G$. Indeed, a G-module homomorphism $\phi : \mathbb{Z} \longrightarrow A$ is uniquely determined by $\phi(1)$ which must be in A^G because $\mathbb{Z}$ is a trivial G-module. Therefore $\phi \mapsto \phi(1)$ is an isomorphism $\mathrm{Hom}_G(\mathbb{Z}, A) \longrightarrow A^G$.

Since $\mathrm{Hom}_G(\mathbb{Z}, -)$ is left exact, we see immediately that the functor $A \mapsto A^G$ of invariants is left exact. Or prove this directly! We may use the resolution F to compute the derived functors, which are $H^q(G, A)$. In other words, $H^q(G, A) = \mathrm{Ext}_q(\mathbb{Z}, A)$. Now (a) is clear.

For (b), an element of $\phi \in \mathrm{Hom}(F^1, A)$ is determined by its values on the basis elements (x) of F, with $x \in G$. Thus we may think of f as just a map $G \longrightarrow A$. The differential $d : F^2 \longrightarrow F^1$ maps (x, y) to $x(y) - (xy) + (x)$ by the formula in Exercise 3. Therefore $df = 0$ in $\mathrm{Hom}_G(F^2, A)$ if and only if $xf(y) - f(xy) + f(x) = 0$, which we recognize as the *crossed-homomorphism* property (4). These are the cocycles $Z^1(G, A)$. To identify the coboundaries, observe that the unique basis element of F^0 is the empty sequence $\epsilon = ()$. The map $d_i : F^1 \longrightarrow F^0$ is the map

$$df(x) = x\epsilon - \epsilon.$$

Thus to $\alpha \in \mathrm{Hom}(F^0, A)$ we may associate the element $a = \alpha(\epsilon)$, and we have $d_i\alpha(x) = xa - a$.

Finally for (c), the map $d : F^3 \longrightarrow F^2$ maps

$$(x, y, z) \to x(y, z) - (xy, z) + (x, yz) - (x, y).$$

Hence for $f \in \mathrm{Hom}(F^2, A)$ to be a cocycle, the condition $df = 0$ is exactly the cocycle condition (5). We see in the same way that the map $f : G \times G \longrightarrow A$ is a coboundary if and only if it has the form

$$f(x, y) = xh(y) - h(xy) + h(x) \tag{6}$$

for some map $h : G \longrightarrow A$. This completes the solution to Problem 4.

For the following exercise we switch to multiplicative notation. Thus the cocycle condition (5) may be rewritten

$$f(xy, z)f(x, y) = {}^x f(y, z)f(x, yz). \tag{7}$$

The cocycle is called *normalized* if $f(1, 1) = 1$.

Proposition 1. *Every cocycle is equivalent to a normalized one. If f is a normalized cocycle then for all $t \in G$ we have*

$$f(1,t) = f(t,1) = 1, \qquad f(t,t^{-1}) = {}^t f(t^{-1},t). \tag{8}$$

Proof. Cocycles are equivalent if they differ by a coboundary, which can be an arbitrary function of the form

$$f_0(x,y) = {}^x h(y)h(y)/h(xy).$$

We may take h to be the constant function $h(x) = f(1,1)$, so $f_0(1,1) = f(1,1)$. Dividing by this coboundary gives an equivalent normalized cocycle. Now in the cocycle relation (7) take $x = t$, $y = z = 1$ to get $f(t,1)f(t,1) = f(1,1)f(t,1)$ and deduce that $f(t,1) = 1$; and similarly taking $x = y = 1$ and $z = t$ gives $f(1,t) = 1$. Finally we may take $x = z = t$ and $y = t^{-1}$ to get $f(1,t)f(t,t^{-1}) = {}^t f(t^{-1},t)f(t,1)$ so $f(t,t^{-1}) = {}^t f(t^{-1},t)$. □

5. **Group extensions**. *Let W be a group and A a normal subgroup written multiplicatively. We assume that A is abelian. (Lang doesn't assume this at least in some copies of Algebra, but this hypothesis is essential.) Let $G = W/A$ be the factor group. Let $F : G \longrightarrow W$ be a choice of coset representatives. Define*

$$f(x,y) = F(x)F(y)F(xy)^{-1}.$$

(a) Prove that f is A-valued and that $F : G \times G \longrightarrow A$ is a 2-cocycle.

(b) Given a group G and an abelian group A, we view an extension W as an exact sequence

$$1 \longrightarrow A \longrightarrow W \longrightarrow G \longrightarrow 1$$

Show that if two such extensions are isomorphic then the 2-cocycles associated to these extensions define the same class in $H^1(G,A)$.

(c) Prove that the map we obtained above from the isomorphism classes of group extensions to $H^2(G,A)$ is a bijection.

Solution. The action of G on A is by conjugation. Thus in multiplicative notation,

$${}^x a = F(x)aF(x)^{-1}. \tag{9}$$

Note that this action does not depend on the choice of representative $F(x)$ because if we change the representative, we change it by an element of A, and A is assumed abelian.

(a) *A priori*, $f(x, y)$ is an element of W, but if we apply the projection map $\pi : W \longrightarrow G$ to it we get $xy(xy)^{-1} = 1$. Thus $f(x, y)$ is in the kernel of π, which is A. To check the cocycle condition, since we are writing A multiplicatively, and using the fact that A is abelian to move the terms around, we need to check

$$ {}^{x}f(y, z)f(x, yz) = f(x, y)f(xy, z). \tag{10} $$

which is the condition (5) in multiplicative notation. The left-hand side equals

$$ {}^{x}(F(y)F(z)F(yz)^{-1})F(x)F(yz)F(xyz)^{-1}.F(x)F(y)F(xy) $$

By (9) this equals

$$ F(x)F(y)F(z)F(yz)^{-1}F(x)^{-1}\cdot F(x)F(yz)F(xyz)^{-1} = F(x)F(y)F(z)F(xyz)^{-1}. $$

Similarly the right hand side of (10) also equals $F(x)F(y)F(z)F(xyz)^{-1}$ and (10) is proved.

We need to check that the cohomology class of this cocycle does not depend on the choice of "section" $F : G \longrightarrow W$. If we change the section to another one F', then since $F(x)$ and $F'(x)$ both must project back to x we have $F'(x) = \varphi(x)F(x)$ where $\varphi(x) \in A$. Now consider

$$ f'(x, y) = F'(x)F'(y)F'(xy)^{-1}. $$

This equals

$$ \varphi(x)F(x)\varphi(y)F(y)F(xy)\varphi(xy)^{-1} = \varphi(x) \cdot {}^{x}\varphi(y)F(x)F(y)F(xy)\varphi(xy)^{-1} $$

and since $F(x)F(y)F(xy)$ and $\varphi(xy)^{-1}$ are both in A, which is abelian, we may write

$$ f'(x, y) = f_0(x, y)f(x, y), \qquad f_0(x, y) = \varphi(x) \cdot {}^{x}\varphi(y) \cdot \varphi(xy)^{-1}. $$

Comparing this with (6), we see that f_0 is a coboundary, so f and f' differ by a coboundary, and we see that each extension determines a well-defined class in $Z^2(G, A)/B^2(G, A)$. The statement asked in (b), that if two such extensions are isomorphic then the 2-cocycles associated to these extensions define the same class in $H^1(G, A)$ is now clear, because if $1 \longrightarrow A \longrightarrow W' \longrightarrow G \longrightarrow 1$ is an equivalent extension then we may identify W with W', and the only issue is that the section F might change, but we have already checked that this only changes f by a coboundary.

(c) We must complement the construction above by showing how to start with a 2-cocycle f and produce an extension $1 \longrightarrow A \longrightarrow W \longrightarrow G \longrightarrow 1$.

We are permitted to change the 2-cocycle by a coboundary. We will therefore assume that f is normalized. We define W to be the set of ordered pairs (a, x) with $a \in A$ and $x \in G$, with multiplication

$$(a, x)(b, y) = (f(x, y)a \cdot {}^x b, xy).$$

Let us check the associative law. We have

$$((a, x)(b, y))(c, z) = (f(x, y)a \cdot {}^x b, xy)(c, z) = (f(x, y)\, f(xy, z)a \cdot {}^x b \cdot {}^{xy} c, xyz)$$

while

$$(a, x)((b, y)(c, z)) = (a, x)(f(y, z) \cdot b \cdot {}^y c, yz) = (f(x, yz)a \ \cdot {}^x(f(y, z) \cdot b \cdot {}^y c), xyz)\,.$$

This equals

$$(f(x, yz)^x f(y, z)a \cdot {}^x b \cdot {}^{xy} c, xyz).$$

So if f satisfies the cocycle relation (10) we have

$$((a, x)(b, y))(c, z) = (a, x)((b, y)(c, z)),$$

confirming the associative law.

Since f is normalized, it is easy to check that $(1, 1)$ serves as an identity element. We may also check the existence of inverses. Since $(a, x) = (a, 1)(1, x)$ it is sufficient to exhibit inverses for $(a, 1)$ and $(1, x)$ separately. We have $(a, 1)^{-1} = (a^{-1}, 1)$ while

$$(1, x)^{-1} = (f(x^{-1}, x)^{-1}, x^{-1}). \tag{11}$$

Indeed it is straightforward that $(f(x^{-1}, x)^{-1}, x^{-1})(1, x) = (1, 1)$, while the other identity

$$(1, x)(f(x^{-1}, x)^{-1}, x^{-1}) = (1, 1)$$

can be checked using (8) or by exhibiting another right inverse and then remembering that in a group, a left and right multiplicative inverse must coincide.

We have proved that W is a group. The maps $A \to W$ in which $a \mapsto (a, 1)$ and $W \to G$ which is the projection on the second component give us a group extension.

We need to relate this to our original construction. Choose the section $F : G \longrightarrow W$ in which $F(x) = (1, x)$. We will prove that

$$F(x)F(y)F(xy)^{-1} = f(x, y). \tag{12}$$

Note that this will solve (c).

The left-hand side of (12) is

$$(1, x)(1, y)(f((xy)^{-1}, xy)^{-1}, (xy)^{-1}) = (f(x, y), xy)(f((xy)^{-1}, xy)^{-1}, (xy)^{-1}).$$

Using (8) this equals

$$(f(x, y), xy)({}^{(xy)^{-1}}f(xy, (xy)^{-1})^{-1}, (xy)^{-1}) = (f(x, y)f(xy, (xy)^{-1})^{-1}f(xy, (xy)^{-1}), 1)$$

or $(f(x, y), 1)$. Since we are identifying A with its image in W, this proves (12).