Homework 5 Solutions

November 3, 2016

We discussed Tor in the lecture. The functor Ext is similar and is the subject of this homework. While Tor is the derived functor of the right exact functor $\otimes$, Ext is the derived functor of the left exact functor Hom.

In the following I use $\square$ to mark the end of a solution.

Problem 1. The functor $\mathrm{Hom}(A, B)$ is left exact. Since it is contravariant in A, it is worth writing out what this means. Given short exact sequences

$$A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

and

$$0 \longrightarrow B' \xrightarrow{h} B \xrightarrow{k} B''$$

we have short exact sequences

$$0 \longrightarrow \mathrm{Hom}(A'', B) \xrightarrow{g^*} \mathrm{Hom}(A, B) \xrightarrow{f^*} \mathrm{Hom}(A', B)$$

and

$$0 \longrightarrow \mathrm{Hom}(A, B') \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, B'').$$

Prove the exactness of the first.

Solution. If $\rho : C \longrightarrow D$ is any homomorphism, let

$$\rho^* : \mathrm{Hom}(D, B) \longrightarrow \mathrm{Hom}(C, B)$$

be composition with ρ. That is, $\rho^*(\phi) = \phi \circ \rho$ for $\phi : D \longrightarrow B$. First, the injectivity of $g^* : \mathrm{Hom}(A'', B) \longrightarrow \mathrm{Hom}(A, B)$ follows from the surjectivity of g. It is also clear that $f^* \circ g^* = (gf)^* = 0$ since $g \circ f = 0$. What remains to be checked is that if $\varphi \in \mathrm{Hom}(A, B)$ and $f^*(\varphi) = 0$ then φ is in the image of g^*. Indeed, since $\varphi \circ f = 0$, the map φ vanishes on the image of f, which is the kernel of g. We may define a map $\psi : A'' \longrightarrow B$ as follows. Since g is surjective, every element of A'' is of the form $g(a)$ for some $a \in A$, and so we define $\psi(a'') = \varphi(a)$ if $a'' = g(a)$. Since $\ker(g) \subseteq \ker(\varphi)$ this is well-defined and $g^*(\psi) = \psi \circ g = \varphi$, proving exactness. $\square$

All modules are over a fixed commutative ring. Like Tor, there are two definitions of Ext. To define $\mathrm{Ext}(A, B)$, we may start with a projective presentation of A, that is, a short exact sequence $0 \longrightarrow R \longrightarrow P \xrightarrow{\alpha} A \longrightarrow 0$. Alternatively we may start with a short exact sequence $0 \longrightarrow B \xrightarrow{\beta} I \longrightarrow Q \longrightarrow 0$ with I injective. Part of the problem is to show that the two resulting (bi)functors are naturally isomorphic. Until this is established, we have to distinguish the two functors, and we will invent a notation for this. We will define $\mathrm{Ext}(A_\alpha, B)$ to be the cokernel of the natural homomorphism $\mathrm{Hom}(P,B) \longrightarrow \mathrm{Hom}(R,B)$, and we will define $\mathrm{Ext}(A, B_\beta)$ to be the cokernel of the natural homomorphism $\mathrm{Hom}(A,I) \longrightarrow \mathrm{Hom}(A,Q)$.

Problem 2. Suppose that $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is exact. Prove that we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(A,B') \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B'') \longrightarrow \mathrm{Ext}(A_\alpha,B') \longrightarrow$$

$$\mathrm{Ext}(A_\alpha,B) \longrightarrow \mathrm{Ext}(A_\alpha,B'')$$

Solution. Consider the diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(A,B') & \longrightarrow & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A,B'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(P,B') & \longrightarrow & \mathrm{Hom}(P,B) & \longrightarrow & \mathrm{Hom}(P,B'') \\
 & & \downarrow \mu & & \downarrow \nu & & \downarrow \omega \\
0 & \longrightarrow & \mathrm{Hom}(R,B') & \longrightarrow & \mathrm{Hom}(R,B) & \longrightarrow & \mathrm{Hom}(R,B'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Ext}(A_\alpha,B') & \longrightarrow & \mathrm{Ext}(A_\alpha,B) & \longrightarrow & \mathrm{Ext}(A_\alpha,B'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

The rows and columns are exact, so we can identify $\mathrm{Hom}(A,B') = \ker(\mu)$ and $\mathrm{Ext}(A_\alpha,B) = \mathrm{coker}(\mu)$, etc. The statement then follows from the Snake Lemma. $\square$

We consider the diagram

$$0 \quad \longrightarrow \quad \mathrm{Hom}(A_\alpha,B') \quad \longrightarrow \quad \mathrm{Hom}(A_\alpha,B) \quad \longrightarrow \mathrm{Hom}(A_\alpha,B'') \quad \longrightarrow \quad 0$$

Problem 3. Suppose $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact. $0 \longrightarrow R \longrightarrow P \xrightarrow{\alpha} A \longrightarrow 0$. Alternatively we may start with a short exact sequence $0 \longrightarrow B \xrightarrow{\beta} I \longrightarrow Q \longrightarrow 0$ with I injective. Write down a similar seven term exact sequence involving $\mathrm{Ext}(A, B_\beta)$. You do not have to prove it. (Or prove it using at most two words, one of them "Lemma.")

Solution.

$$0 \longrightarrow \mathrm{Hom}(A'',B) \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A',B) \longrightarrow \mathrm{Ext}(A'',B_\beta) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}(A,B_\beta) \longrightarrow \mathrm{Ext}(A',B_\beta). \qquad \text{Snake!} \qquad \square$$

Our next goal is to show that $\mathrm{Ext}(A, B_\beta)$ is functorial and independent of β. Let B and B' be given, with embeddings $\beta : B \longrightarrow I$ and $\beta' : B' \longrightarrow I'$ into injective modules. Suppose $f : B \longrightarrow B'$ is given. Let $Q = I/\beta(B)$ and similarly Q'. Using the injectivity of I we may find maps $\phi : I \longrightarrow I'$ and $\bar{\phi} : Q \longrightarrow Q'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & I & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \phi & & \downarrow \bar{\phi} & & \\ 0 & \longrightarrow & B' & \longrightarrow & I' & \longrightarrow & Q' & \longrightarrow & 0 \end{array} \qquad (1)$$

Thus we get a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A,I) & \longrightarrow & \mathrm{Hom}(A,Q) & \longrightarrow & \mathrm{Ext}(A,B_\beta) \\ & & \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} & & \\ 0 & \longrightarrow & \mathrm{Hom}(A,B') & \longrightarrow & \mathrm{Hom}(A,I') & \longrightarrow & \mathrm{Hom}(A,Q') & \longrightarrow & \mathrm{Ext}(A,B'_{\beta'}) \end{array}$$

The map $\bar{\phi}$ induces a map $\bar{f} : \mathrm{Ext}(A, B_\beta) \longrightarrow \mathrm{Ext}(A, B'_{\beta'})$.

4. (a) Show that this induced map does not depend on the choice of ϕ.

(b) Explain why (and in what sense) this implies that $\mathrm{Ext}(A, B_\beta)$ does not depend on the injective embedding β.

(c) Show that $\mathrm{Ext}(A, B)$ defined by injective embeddings of B is a functor. (It is actually a bifunctor but I'm only asking you to show, with A fixed, that is functorial in B.)

(d) Do the same considerations apply without change for any left exact functor $\mathcal{F}$ instead of the special case $\mathcal{F}B = \mathrm{Hom}(A, B)$?

Solution.

(a) As we saw in the lectures with right exact functors, the key to this uniqueness is the notion of chain homotopy, which we encounter again in another simple form.

Suppose we have two different choices of ϕ, say $\phi = \phi_1$ and ϕ_2. Let $\phi_0 = \phi_1 - \phi_2$. Then since ϕ_1 and ϕ_2 induce the same map f on B, the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\beta} & I & \xrightarrow{p} & Q & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow \phi_0 & \swarrow h & \downarrow \bar{\phi}_0 & & \\ 0 & \longrightarrow & B' & \xrightarrow{\beta'} & I' & \xrightarrow{p'} & Q' & \longrightarrow & 0 \end{array}$$

where the map $h : Q \longrightarrow I'$ has to be explained. The map p is surjective, so every element q of Q equals $p(x)$ for some $x \in I$. We can define $h(q) = \phi_0(x)$ provided we check that this is well-defined. If $p(x') = q$ also, then $x - x'$ are in the kernel of p, hence $x - x' = \beta(b)$ for some b, and since $\phi_0 \circ \beta$ is the zero map we have $\phi_0(x) = \phi_0(x')$. Thus there is a well-defined map h such that $h \circ p = \phi_0$.

Now consider:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A,I) & \longrightarrow & \mathrm{Hom}(A,Q) & \xrightarrow{\pi} & \mathrm{Ext}(A,B_\beta) \\ & & \downarrow 0 & & \downarrow (\phi_0)_* & \swarrow h_* & \downarrow (\overline{\phi_0})_* & & \downarrow t \\ 0 & \longrightarrow & \mathrm{Hom}(A,B') & \longrightarrow & \mathrm{Hom}(A,I') & \xrightarrow{p'_*} & \mathrm{Hom}(A,Q') & \xrightarrow{\pi'} & \mathrm{Ext}(A,B'_{\beta'}) \end{array}$$

Here if $\psi : X \longrightarrow Y$ is any map, I'm denoting by $\psi_* : \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(B, X)$ the map induced by composition with ψ; that is, $\psi_*(f) = \psi \circ f$ for $f \in \mathrm{Hom}(A, X)$. We are required to check that the induced map $\mathrm{Ext}(A, B_\beta) \longrightarrow \mathrm{Ext}(A, B'_\beta)$ is zero. Indeed, denoting this map t, we see that $t\pi = \pi'(\overline{\phi_0})_* = \pi' \circ p'_* \circ h_* = 0$. $\square$.

(b) Suppose we choose another injective embedding $\beta' : B \longrightarrow I'$. Applying the above construction to the identity map $B \longrightarrow B$ gives us an induced map $\mathrm{Ext}(A, B_\beta) \longrightarrow \mathrm{Ext}(A, B_{\beta'})$. Similarly we have an induced map $\mathrm{Ext}(A, B_{\beta'}) \longrightarrow \mathrm{Ext}(A, B_\beta)$. We will argue that these are inverse isomorphisms.

Let $i : B \longrightarrow I$ and $i' : B \longrightarrow I'$ be injective embeddings. Using the injectivity of I, extend i' to a map $\phi : I \longrightarrow I'$, and using the injectivity of I' extend i to a map $\psi : I' \longrightarrow I$. We do *not* know that ϕ and ψ are inverses of each other; they probably are not. However we let

$$f : \mathrm{Ext}(A, B_\beta) \longrightarrow \mathrm{Ext}(A, B'_{\beta'}) \qquad \text{and} \qquad g : \mathrm{Ext}(A, B'_{\beta'}) \longrightarrow \mathrm{Ext}(A, B_\beta)$$

be the induced maps. These we will argue *are* inverse homomorphisms. To show that $gf = 1_{\mathrm{Ext}(A,B_\beta)}$ we use part (a) as follows. We have two different maps $I \longrightarrow I$ that make the following diagram commute, namely either $\psi\phi$ or 1_I:

$$\begin{array}{ccccc}
0 & \longrightarrow & B & \xrightarrow{\beta} & I \\
 & & \downarrow 1_B & & \downarrow \\
0 & \longrightarrow & B & \xrightarrow{\beta} & I
\end{array}$$

We use these in place of ϕ, ϕ' in (a) with $B' = B$. Since by (a) they induce the same endomorphism of $\mathrm{Ext}(A, B_\beta)$ we conclude that $gf = 1_{\mathrm{Ext}(A,B_\beta)}$ and similarly $fg = 1_{\mathrm{Ext}(A,B_{\beta'})}$. $\square$

(c) We will describe a functor $\mathcal{F}$ from R-modules to R-modules such that $\mathcal{F}B \cong \mathrm{Ext}(A, B)$ for all B.

For each object B in the category of R-modules we choose and fix an injective embedding $\beta : B \longrightarrow I$. (To avoid set theoretic difficulties, this should be done in a canonical way, but this is a secondary detail.)

Then we can define $\mathcal{F} = \mathrm{Ext}(A, B_\beta)$.

If $f : B \longrightarrow B'$ is any homomorphism, then by (a) there is a unique homomorphism $\mathcal{F}f$ making the following diagram commute, where $\phi : I \longrightarrow I'$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(A, B) & \longrightarrow & \mathrm{Hom}(A, I) & \longrightarrow & \mathrm{Hom}(A, Q) & \xrightarrow{\pi} & \mathrm{Ext}(A, B_\beta) \\
 & & \downarrow f_* & & \downarrow \phi_* & & \downarrow & & \downarrow \mathcal{F}f \\
0 & \longrightarrow & \mathrm{Hom}(A, B') & \longrightarrow & \mathrm{Hom}(A, I') & \xrightarrow{p'_*} & \mathrm{Hom}(A, Q') & \xrightarrow{\pi'} & \mathrm{Ext}(A, B'_{\beta'})
\end{array}$$

To verify that $\mathcal{F}$ is a functor, let us show that if $g : B' \longrightarrow B''$ is another homomorphism then $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(A, B) & \longrightarrow & \mathrm{Hom}(A, I) & \longrightarrow & \mathrm{Hom}(A, Q) & \xrightarrow{\pi} & \mathrm{Ext}(A, B_\beta) \\
 & & \downarrow (g \circ f)_* & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(A, B'') & \longrightarrow & \mathrm{Hom}(A, I'') & \xrightarrow{p'_*} & \mathrm{Hom}(A, Q'') & \xrightarrow{\pi'} & \mathrm{Ext}(A, B''_{\beta''})
\end{array}$$

Both maps $\mathcal{F}(g \circ f)$ and $\mathcal{F}g \circ \mathcal{F}f$ make the diagram commute, so by the uniqueness in (a), they are the same map. $\square$

(d) Yes.

We proved the following in class. Let

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

be exact. Find presentations $0 \longrightarrow R' \longrightarrow P' \xrightarrow{\alpha'} A' \longrightarrow 0$ and $0 \longrightarrow R'' \longrightarrow P'' \xrightarrow{\alpha''} A'' \longrightarrow 0$ with P' and P'' projective. Let P be $P' \oplus P''$ and consider the short exact sequence

$$0 \longrightarrow P' \xrightarrow{i} P \xrightarrow{p} P'' \longrightarrow 0$$

where i is the projection and p is the projection. Then we may find a map $\alpha : P \longrightarrow A$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \xrightarrow{i} & P & \xrightarrow{p} & P & \longrightarrow & 0 \\
 & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\
0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0
\end{array}$$

Then of course we can complete this to a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R' & \xrightarrow{i} & R & \xrightarrow{p} & R'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \xrightarrow{i} & P & \xrightarrow{p} & P'' & \longrightarrow & 0 \\
 & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\
0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

5. Dualize this result: start with

$$0 \longrightarrow B' \xrightarrow{h} B \xrightarrow{k} B'' \longrightarrow 0$$

and injective embeddings $0 \longrightarrow B' \xrightarrow{\beta'} I' \longrightarrow Q' \longrightarrow 0$ and $0 \longrightarrow B'' \xrightarrow{\beta''} I'' \longrightarrow Q'' \longrightarrow 0$. State and prove the corresponding result.

Solution. You must show that if $I = I' \oplus I''$ then we can complete the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \xrightarrow{i} & B & \xrightarrow{p} & B'' & \longrightarrow & 0 \\
 & & \downarrow \beta' & & \downarrow \beta & & \downarrow \beta'' & & \\
0 & \longrightarrow & I' & \xrightarrow{i} & I & \xrightarrow{p} & I'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q' & \xrightarrow{f} & Q & \xrightarrow{g} & Q'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The map $\beta : B \longrightarrow I$ must be constructed. Using the injectivity of I', there exists a map $\beta_1 : B \longrightarrow I'$ such that $\beta_1 \circ i = \beta'$. Also, let $\beta_2 : B \longrightarrow I''$ be the composition $\beta'' \circ p$. Then we may define $\beta(b) = (\beta_1(b), \beta_2(b))$ and it is easy to see that the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \xrightarrow{i} & B & \xrightarrow{p} & B'' & \longrightarrow & 0 \\
 & & \downarrow \beta' & & \downarrow \beta & & \downarrow \beta'' & & \\
0 & \longrightarrow & I' & \xrightarrow{i} & I & \xrightarrow{p} & I'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

commutes. We have induced maps $I' \longrightarrow Q'$ etc completing the picture, just as in the Snake lemma. $\square$

6. (Lambek) We sketched a proof of the following fact in class. Consider a commutative diagram with exact rows:

$$\begin{array}{ccccc}
A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\
\downarrow f' & \Sigma_1 & \downarrow f & \Sigma_2 & \downarrow f'' \\
B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B''
\end{array}$$

Define

$$\operatorname{coker}(\Sigma_1) = (\operatorname{im}(f) \cap \operatorname{im}(\beta_1))/\operatorname{im}(f\alpha_1),$$

$$\ker(\Sigma_2) = \ker(\beta_2 f)/((\ker(\alpha_2) + \ker(f)).$$

Show $\operatorname{coker}(\Sigma_1) \cong \ker(\Sigma_2)$.

Solution. The image of $\ker(\beta_2 f)$ in B under the map f clearly is contained in both $\operatorname{im}(f)$ and $\ker(\beta_2) = \operatorname{im}(\beta_1)$. We can thus compose this map with the projection onto $\ker(\Sigma_2)$:

$$\ker(\beta_2 f) \xrightarrow{f} \operatorname{im}(f) \cap \operatorname{im}(\beta_1) \xrightarrow{p} (\operatorname{im}(f) \cap \operatorname{im}(\beta_1))/\operatorname{im}(f\alpha_1) \tag{2}$$

It is sufficient to show that this composition is surjective, and that its kernel is exactly $\ker(\alpha_2) + \ker(f)$.

Surjective: suppose that $b \in \operatorname{im}(f) \cap \operatorname{im}(\beta_1)$. Then we may write $b = f(a) = \beta_1(b')$ with $a \in A$ and $b' \in B'$. Since $\beta_2 f(a) = \beta_2\beta_1(b') = 0$ we have $f \in \ker(\beta_2 f)$. This proves that the composition (2) is surjective.

Finally we must show that the kernel is $\ker(\alpha_2) + \ker(f)$. It is clear that $\ker(f)$ is contained in the kernel of (2), and if $a \in \ker(\alpha_2) = \operatorname{im}(\alpha_1)$, then $f(a) \in \operatorname{im}(f\alpha_1)$. Thus $\ker(\alpha_2)$ is also contained in the kernel of (2). On the other hand, suppose that $a \in \ker(\beta_2 f)$ is in the kernel of (2). This means that $f(a) \in \operatorname{im}(f\alpha_1)$, so we may write $f(a) = f\alpha_1(a')$ for some $a' \in A'$. Now write $a = a_1 + a_2$ where $a_1 = \alpha_1(a')$ and $a_2 = a - \alpha_1(a')$. Then $a_1 \in \operatorname{im}(\alpha_1) = \ker(\alpha_2)$, while $a_2 \in \ker(f)$.

We have proved that the composition (2) is surjective and its kernel is exactly $\ker(\alpha_2) + \ker(f)$. Therefore

$$(\operatorname{im}(f) \cap \operatorname{im}(\beta_1))/\operatorname{im}(f\alpha_1) \quad \cong \quad \ker(\beta_2 f)/((\ker(\alpha_2) + \ker(f)).\square$$

7. Use Lambek's Lemma to prove that $\operatorname{Ext}(A_\alpha, B)$ and $\operatorname{Ext}(A, B_\beta)$ are isomorphic.

First Solution. Recall that $0 \longrightarrow R \longrightarrow P \xrightarrow{\alpha} A \longrightarrow 0$ is a projective resolution and $0 \longrightarrow B \xrightarrow{\beta} I \longrightarrow Q \longrightarrow 0$ is an injective embedding. Consider:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A,I) & \longrightarrow & \mathrm{Hom}(A,Q) & \longrightarrow & \mathrm{Ext}(A,B_\beta) \\
 & & \downarrow & & \downarrow & \Sigma_4 & \downarrow & \Sigma_5 & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(P,B) & \longrightarrow & \mathrm{Hom}(P,I) & \longrightarrow & \mathrm{Hom}(P,Q) & \longrightarrow & 0 \\
 & & \downarrow & \Sigma_2 & \downarrow & \Sigma_3 & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(R,B) & \longrightarrow & \mathrm{Hom}(R,I) & \longrightarrow & \mathrm{Hom}(R,Q) & & \\
 & & \downarrow & \Sigma_1 & \downarrow & & & & \\
 & & \mathrm{Ext}(A_\alpha,B) & \longrightarrow & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

We claim that the rows and columns are exact. Mostly this follows from the left exactness of Hom and the definitions of $\mathrm{Ext}(A_\alpha, B)$ and $\mathrm{Ext}(A, B_\beta)$ as the cokernel and kernel of the middle maps in the first column and first row. The surjectivity of the two maps

$$\mathrm{Hom}(P,I) \longrightarrow \mathrm{Hom}(R,I), \qquad \mathrm{Hom}(P,I) \longrightarrow \mathrm{Hom}(P,Q)$$

follows from the injectivity of I and the projectivity of P.

Now with the notation introduced with Lambek's Lemma,

$$\ker(\Sigma_1) \cong \mathrm{coker}(\Sigma_2) \cong \ker(\Sigma_3) \cong \mathrm{coker}(\Sigma_4) \cong \ker(\Sigma_5).$$

It is easy to check that the kernel of Σ_1 is all of $\mathrm{Hom}(R, B)$ modulo the kernel of the map $\mathrm{Hom}(R, B) \longrightarrow \mathrm{Ext}(A_\alpha, B)$, that is $\ker(\Sigma_1) \cong \mathrm{Ext}(A_\alpha, B)$ and similarly $\ker(\Sigma_5) \cong \mathrm{Ext}(A, B_\beta)$. $\square$

Second Solution This solution does not use Lambek's Lemma. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A,I) & \longrightarrow & \mathrm{Hom}(A,Q) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P,B) & \longrightarrow & \mathrm{Hom}(P,I) & \longrightarrow & \mathrm{Hom}(P,Q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(R,B) & \longrightarrow & \mathrm{Hom}(R,I) & \longrightarrow & \mathrm{Hom}(R,Q) & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Ext}(A_\alpha,B) & \longrightarrow & 0 & & & &
\end{array}$$

The kernel of the map $\mathrm{Hom}(P, X) \to \mathrm{Hom}(R, X)$ is $\mathrm{Hom}(A, X)$, where $X = B, I, Q$, and the cokernel of the map $\mathrm{Hom}(P, B) \to \mathrm{Hom}(R, B)$ is $\mathrm{Ext}(A_\alpha, B)$. So the Snake Lemma gives an exact sequence:

$$0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,I) \to \mathrm{Hom}(A,Q) \to \mathrm{Ext}(A_\alpha,B) \to 0.$$

But this is the definition of $\mathrm{Ext}(A, B_\alpha)$, so

$$\mathrm{Ext}(A_\alpha, B) \cong \mathrm{Ext}(A, B_\alpha).$$