Homework 4 Solutions

From Lang's *Algebra*. You were asked to do Problems 9 and 14 in Chapter 3. I'll only partially do Problem 9.

Chapter 3, Problem 9. (a) Let A be a commutative ring and let M be an A-module. Let S be a multiplicative subset of A. Define $S^{-1}M$ in a manner analogous to the one we used to define $S^{-1}A$, and show that $S^{-1}M$ is an $S^{-1}A$-module.

(b) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence show that

$$0 \longrightarrow S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M'' \longrightarrow 0$$

is exact.

Partial Solution. For (a), define $S^{-1}M$ to be pairs m/s with $m \in M$ and $s \in S$ modulo the equivalence relation $m/s = m'/s'$ if and only if $tsm - ts'm = 0$ for some $t \in S$. There are some things to check but they are straightforward.

For (b), it is sufficient to show that if $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact then so is

$$S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M''.$$

It is clear that the composition map $S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M''$ is zero, so it is enough to show that if $m/s \in S^{-1}M$ maps to zero in $S^{-1}M'$ then $m/s = S^{-1}f(x)$ for some $x \in S^{-1}M'$. Indeed, since $S^{-1}g(m/s) = 0$ there exists some $t \in S$ such that $tg(m) = 0$. Therefore $tm \in \ker(g) = \mathrm{im}(f)$, and we can write $tm = f(m')$ for some $m' \in M$. Now

$$S^{-1}f(m'/st) = f(m')/st = m/s$$

so we may take $x = m'/st$.

We remark concerning Problem 9 that actually $S^{-1}M \cong S^{-1}A \otimes M$ and so this problem says that $S^{-1}A$ is a flat A-module. See Proposition 3.2 on page 13.

Chapter 16, Problem 14. From the snake Lemma, the following sequence is exact.

$$\ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h).$$

For (i), assuming f and h are monomorphisms (injective homomorphisms) $\ker(f) = \ker(h) = 0$. If we have an exact sequence $0 \to M \to 0$ then $M = 0$, so $\ker(g) = 0$, proving (i). Part (ii) is similar noting that f is surjective if and only of $\operatorname{coker}(f) = 0$. Finally for (iii), the assumption implies that we have an exact sequence

$$0 \to \ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h) \to 0.$$

With this extra information we may now argue similarly to (i) and (ii).

Chapter 16, Problem 4. *Let $\varphi : A \to B$ be a commutative ring homomorphism. Let E be an A-module and F a B-module. Let F_A be the A-module obtained from F via the operation of A on F through φ, that is for $y \in F_A$ and $a \in A$ this operation is given by*

$$(a, y) \mapsto \varphi(a)y.$$

Show that there is a natural isomorphism

$$\operatorname{Hom}_B(B \otimes_A E, F) \cong \operatorname{Hom}_A(E, F_A). \tag{1}$$

Remark. Extension and restriction of scalars are functors between the categories of A-modules and B-modules.

A-modules	$\longrightarrow$	B-modules	
E	$\longmapsto$	$B \otimes_A E$	(extension)
F_A	$\longleftarrow$	F	(restriction)

The isomorphism (1) is analogous to the relationship between two adjoint operators between Hilbert spaces. Thus if $\boldsymbol{H}, \boldsymbol{H}'$ are Hilbert spaces and $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}'$ is an operator we have

$$\langle Tv, v' \rangle = \langle v, T^* v' \rangle.$$

In view of this analogy we say that extension and restriction of scalars are *adjoint functors*.

Solution. Let $\alpha : E \longrightarrow F_A$ be an A-module homomorphism. The map $(b, x) \mapsto b\alpha(x)$ from $B \times E \longrightarrow F$ is A-bilinear so it induces a homomorphism of A-modules $\beta : B \otimes_A E \longrightarrow F$ such that $\beta(b \otimes x) = b\alpha(x)$. But this map is actually B-linear if we remember how $B \otimes_A E$ is made into a B-module (Lang, p. 623). Indeed,

$$\beta(b_1 \cdot (b \otimes x)) = \beta(b_1 b \otimes x) = b_1 b\alpha(x) = b_1 \cdot \beta(b \otimes x).$$

Since $B \otimes_A E$ is generated by elements of the form $b \otimes x$, this confirms that β is a B-module homomorphism. Now the map $\alpha \mapsto \beta$ gives us a map

$$\mathrm{Hom}_A(E, F_A) \longrightarrow \mathrm{Hom}_B(B \otimes_A E, F). \tag{2}$$

To construct a map in the other direction, observe that $j : E \longrightarrow B \otimes_A E$ defined by $j(x) = 1 \otimes x$ is an A-module homomorphism. Composition with j gives us a map

$$\mathrm{Hom}_B(B \otimes_A E, F) \longrightarrow \mathrm{Hom}_A(E, F_A). \tag{3}$$

That is, given $\beta \in \mathrm{Hom}_B(B \otimes_A E, F)$ define $\alpha \in \mathrm{Hom}_A(E, F_A)$ to be the composition $\beta \circ j$.

We must show that these two constructions are inverses of each other. Thus suppose we start with $\beta \in \mathrm{Hom}_B(B \otimes_A E, F)$, then define $\alpha = \beta \circ j$. We must show that β is recovered from α by the map (2). It is sufficient to show that $\beta(b \otimes x) = b\alpha(x)$ since $B \otimes_A E$ is generated as an A-module by elements of the form $b \otimes x$. Indeed since β is B-linear we have

$$\beta(b \otimes x) = \beta(b \cdot (1 \otimes x)) = b\,\beta(1 \otimes x) = b\beta j(x) = b\alpha(x).$$

This proves that the composition

$$\mathrm{Hom}_B(B \otimes_A E, F) \longrightarrow \mathrm{Hom}_A(E, F_A) \longrightarrow \mathrm{Hom}_B(B \otimes_A E, F)$$

is the identity map. As for the composition

$$\mathrm{Hom}_A(E, F_A) \longrightarrow \mathrm{Hom}_B(B \otimes_A E, F) \longrightarrow \mathrm{Hom}_A(E, F_A)$$

if we start with $\alpha \in \mathrm{Hom}_A(E, F_A)$ and define β by $\beta(b \otimes x) = b\alpha(x)$ then

$$\beta \circ j(x) = \beta(1 \otimes x) = 1 \cdot \alpha(x),$$

so this composition is also the identity.

Chapter 16, Problem 6. If M, N are flat so is $M \otimes N$.

Solution: If $0 \longrightarrow T \longrightarrow U$ is exact consider the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & (M \otimes N) \otimes T & \longrightarrow & (M \otimes N \otimes U) \\ & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & M \otimes (N \otimes T) & \longrightarrow & M \otimes (N \otimes U) \end{array}$$

We are using the naturality of the isomorphism $(M \otimes N) \circ T \longrightarrow M \otimes (N \otimes T)$. Since N is flat $N \otimes T \longrightarrow N \otimes U$ is injective, and then since M is flat, the bottom row is exact. Hence the top row is exact proving that $M \otimes N$ is flat.

Chapter 16, Problem 7. Let F be a flat R-module and let $a \in R$ be an element that is not a zero divisor. If $ax = 0$ for some $x \in R$ then $x = 0$.

Solution.

Lemma 1. *We have a natural isomorphism $R \otimes M \cong M$.*

Note that this is a special case of Proposition 2.7 on page 612. It is a basic property of the tensor product and probably deserves more prominence in the book!

Proof. The multiplication map $R \times M \longrightarrow M$ is bilinear, so there is a homomorphism $R \otimes M \longrightarrow M$ such that $r \otimes m \longmapsto rm$. On the other hand, $m \mapsto 1 \otimes m$ is a homomorphism $M \longrightarrow R \otimes M$. It is easy to see that these maps are inverses of each other. Naturality means that if $f : M \longrightarrow N$ is a homomorphism the diagram

$$\begin{array}{ccc} R \otimes M & \longrightarrow & M \\ \downarrow & & \downarrow \\ R \otimes N & \longrightarrow & N \end{array}$$

commutes, and this is easy to check. □

Now consider the map $f : R \longrightarrow R$ defined by $f(x) = ax$. Since R is commutative this is a homomorphism, and since a is not a zero divisor, it is injective. Tensoring with F and using the Lemma, the map $F \longrightarrow F$ which is multiplication by a is injective.

Chapter 16, Problem 8. (i) If S is a multiplicative subset of the commutative ring R. Then $S^{-1}R$ is flat over R.

(ii) M is flat over R if and only if $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R.

(iii) If R is a principal ideal domain, then the module F is flat if and only if it is torsion-free.

Solutions:

Lemma 2. *The $S^{-1}R$ module $S^{-1}M$ is isomorphic to $S^{-1}R \otimes_R M$.*

Proof. We have a homomorphism $S^{-1}M \longrightarrow S^{-1}R \otimes_R M$ that satisfies $m/s \mapsto 1/s \otimes m$, and an inverse homomorphism $S^{-1}R \otimes_R M \longrightarrow S^{-1}M$ that satisfies $a/s \otimes m \mapsto am/s$. We leave the reader to check that these are inverse isomorphisms. □

Now we recognize the flatness of $S^{-1}R$ as being equivalent to Chapter 3 Problem 9b, solved above. This proves (i).

Lemma 3. *Suppose that M is an R-module and N is an $S^{-1}R$-module. Then*

$$S^{-1}M \otimes_{S^{-1}R} N \cong M \otimes_R N.$$

Proof. We make use of the isomorphism

$$S^{-1}M \otimes_{S^{-1}R} N \cong (S^{-1}R \otimes_R M) \otimes_{S^{-1}R} N$$

where in the second tensor product it is understood that the $S^{-1}R$-module structure on $S^{-1}R \otimes_R M$ comes from the action of $S^{-1}R$ on $S^{-1}R$. Using the commutativity and associativity of the tensor product this is isomorphic to

$$(M \otimes_R S^{-1}R) \otimes_{S^{-1}R} N \cong M \otimes_R (S^{-1}R \otimes_{S^{-1}R} N).$$

Now we may use Lemma 1. □

Lemma 4. *Suppose that N is a flat $S^{-1}R$-module. Then it is flat as an R-module.*

Proof. Suppose that $M \longrightarrow M'$ is an injective homomorphism of R-modules. Then $S^{-1}M \longrightarrow S^{-1}M'$ is injective by Chapter 3 Problem 9, solved above. Now since N is flat as an R-module, the map

$$S^{-1}M \otimes_{S^{-1}R} N \longrightarrow S^{-1}M' \otimes_{S^{-1}R} N$$

is injective. But by Lemma 3, this may be interpreted as the natural map $M \otimes_R N \longrightarrow M' \otimes_R N$. So N is flat as an R-module. □

The following fact is part of Problem 10 in Chapter 3. (It was not assigned, but I'll prove it.)

Lemma 5. *Let M be an R-module. Then $M = 0$ if and only if $M_{\mathfrak{p}} = 0$ for all maximal ideals $\mathfrak{p}$ of M.*

Proof. Suppose that a is a nonzero element of M. Let $\mathfrak{a} = \{x \in R | ax = 0\}$. Then $\mathfrak{a}$ is a proper ideal since $1 \notin \mathfrak{a}$ so $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{p}$. We claim that $M_{\mathfrak{p}} \neq 0$. Indeed, if $x/1 \in M_{\mathfrak{p}}$ is zero, then by definition of the localization $M_{\mathfrak{p}}$ we must have $sx = 0$ for some $s \in S = R - \mathfrak{p}$. But this is a contradiction since $s \notin \mathfrak{a}$. □

Now we can do (ii). If M is flat then since $M_{\mathfrak{p}} = S^{-1}R \otimes M$ with $S = R - \mathfrak{p}$, $M_{\mathfrak{p}}$ is a tensor product of two flat R-modules, hence flat.

Conversely, suppose that $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for all $\mathfrak{p}$. Then it is flat as an R-module by Lemma 4. Suppose that

$$A \longrightarrow B$$

is an injective homomorphism of R-modules. We want to show that

$$M \otimes_R A \longrightarrow M \otimes_R B$$

is injective. If not, let K be the kernel, so the following is exact:

$$0 \longrightarrow K \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B.$$

By Lemma 5, if $K \neq 0$ there exists a $\mathfrak{p}$ such that $K_{\mathfrak{p}} \neq 0$. Now by the exactness of localization (with $S = R - \mathfrak{p}$)

$$0 \longrightarrow K_{\mathfrak{p}} \longrightarrow S^{-1}(M \otimes_R A) \longrightarrow S^{-1}(M \otimes_R B)$$

is exact and so $S^{-1}(M \otimes A) \longrightarrow S^{-1}(M \otimes B)$ is not injective. But

$$S^{-1}(M \otimes_R A) \cong S^{-1}R \otimes_R M \otimes_R A \cong M_{\mathfrak{p}} \otimes A$$

and so $M_{\mathfrak{p}} \otimes_R A \longrightarrow M_{\mathfrak{p}} \otimes_R B$ is not injective. But $M_{\mathfrak{p}}$ is flat as an R-module by Lemma 4. This is a contradiction.

(iii) Use Proposition 3.7 on page 618.