Math 210A Homework 5

Due Tuesday, November 1, 2016.

We discussed Tor in the lecture. The functor Ext is similar and is the subject of this homework. While Tor is the derived functor of the right exact functor $\otimes$, Ext is the derived functor of the left exact functor Hom.

1. The functor $\operatorname{Hom}(A, B)$ is left exact. Since it is contravariant in A, it is worth writing out what this means. Given short exact sequences

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B''$$

we have short exact sequences

$$0 \longrightarrow \operatorname{Hom}(A'', B) \longrightarrow \operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(A', B)$$

and

$$0 \longrightarrow \operatorname{Hom}(A, B') \longrightarrow \operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(A, B'').$$

Prove the exactness of the first.

All modules are over a fixed commutative ring. Like Tor, there are two definitions of Ext. To define $\operatorname{Ext}(A, B)$, we may start with a projective presentation of A, that is, a short exact sequence $0 \longrightarrow R \longrightarrow P \xrightarrow{\alpha} A \longrightarrow 0$. Alternatively we may start with a short exact sequence $0 \longrightarrow B \xrightarrow{\beta} I \longrightarrow Q \longrightarrow 0$ with I injective. Part of the problem is to show that the two resulting (bi)functors are naturally isomorphic. Until this is established, we have to distinguish the two functors, and we will invent a notation for this. We will define $\operatorname{Ext}(A_\alpha, B)$ to be the cokernel of the natural homomorphism $\operatorname{Hom}(P, B) \longrightarrow \operatorname{Hom}(R, B)$, and we will define $\operatorname{Ext}(A, B_\beta)$ to be the cokernel of the natural homomorphism $\operatorname{Hom}(A, I) \longrightarrow \operatorname{Hom}(A, Q)$.

2. Suppose that $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is exact. Prove that we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(A, B') \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, B'') \longrightarrow \mathrm{Ext}(A_\alpha, B') \longrightarrow$$

$$\mathrm{Ext}(A_\alpha, B) \longrightarrow \mathrm{Ext}(A_\alpha, B'')$$

3. Suppose $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact. Write down a similar seven term exact sequence involving $\mathrm{Ext}(A, B_\beta)$. You do not have to prove it. (Or prove it using at most two words, one of them "Lemma.")

Our next goal is to show that $\mathrm{Ext}(A, B_\beta)$ is functorial and independent of β. Let B and B be given, with embeddings $\beta : B \longrightarrow I$ and $\beta' : B' \longrightarrow I'$ into injective modules. Suppose $f : B \longrightarrow B'$ is given. Let $Q = I/\beta(B)$ and similarly Q'. Using the injectivity of I we may find maps $\phi : I \longrightarrow I'$ and $\bar{\phi} : Q \longrightarrow Q'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & I & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} & & \\
0 & \longrightarrow & B' & \longrightarrow & I' & \longrightarrow & Q' & \longrightarrow & 0
\end{array}$$

Thus we get a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(A, B) & \longrightarrow & \mathrm{Hom}(A, I) & \longrightarrow & \mathrm{Hom}(A, Q) & \longrightarrow & \mathrm{Ext}(A, B_\beta) \\
 & & \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} & & \\
0 & \longrightarrow & \mathrm{Hom}(A, B') & \longrightarrow & \mathrm{Hom}(A, I') & \longrightarrow & \mathrm{Hom}(A, Q') & \longrightarrow & \mathrm{Ext}(A, B'_{\beta'})
\end{array}$$

This induces a map $\mathrm{Ext}(A, B_\beta) \longrightarrow \mathrm{Ext}(A, B'_{\beta'})$.

4. (a) Show that this induced map does not depend on the choice of ϕ.

(b) Explain why (and in what sense) this implies that $\mathrm{Ext}(A, B_\beta)$ does not depend on the injective embedding β.

(c) Show that $\mathrm{Ext}(A, B)$ defined by injective embeddings of B is a functor. (It is actually a bifunctor but I'm only asking you to show, with A fixed, that is functorial in B.)

(d) Do the same considerations apply without change for any left exact functor $\mathcal{F}$ instead of the special case $\mathcal{F}B = \mathrm{Hom}(A, B)$?

On Wednesday, October 26, we proved the following in class. Let

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

be exact. Find presentations $0 \longrightarrow R' \longrightarrow P' \xrightarrow{\alpha'} A' \longrightarrow 0$ and $0 \longrightarrow R'' \longrightarrow P'' \xrightarrow{\alpha''} A'' \longrightarrow 0$ with P' and P'' projective. Let P be $P' \oplus P''$ and consider the short exact sequence

$$0 \longrightarrow P' \xrightarrow{i} P \xrightarrow{p} P'' \longrightarrow 0$$

where i is the projection and p is the projection. Then we may find a map $\alpha : P \longrightarrow A$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \xrightarrow{i} & P & \xrightarrow{p} & P & \longrightarrow & 0 \\
 & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\
0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0
\end{array}$$

Then of course we can complete this to a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R' & \xrightarrow{i} & R & \xrightarrow{p} & R'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \xrightarrow{i} & P & \xrightarrow{p} & P'' & \longrightarrow & 0 \\
 & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\
0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

5. Dualize this result: start with

$$0 \longrightarrow B' \xrightarrow{h} B \xrightarrow{k} B'' \longrightarrow 0$$

and injective embeddings $0 \longrightarrow B' \xrightarrow{\beta'} I' \longrightarrow Q' \longrightarrow 0$ and $0 \longrightarrow B'' \xrightarrow{\beta''} I'' \longrightarrow Q'' \longrightarrow 0$. State and prove the corresponding result.

6. (Lambek) Consider a commutative diagram with exact rows:

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\ \downarrow f' & \Sigma_1 & \downarrow f & \Sigma_2 & \downarrow f'' \\ B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \end{array}$$

Define

$$\operatorname{coker}(\Sigma_1) = (\operatorname{im}(f) \cap \operatorname{im}(\beta_1)) / \operatorname{im}(f\alpha_1),$$

$$\ker(\Sigma_2) = \ker(\beta_2 f)/((\ker(\alpha_2) + \ker(f)).$$

Show $\operatorname{coker}(\Sigma_1) \cong \ker(\Sigma_2)$.

7. Use Lambek's Lemma to prove that $\operatorname{Ext}(A_\alpha, B)$ and $\operatorname{Ext}(A, B_\beta)$ are isomorphic.