Math 210A Homework 3

Due Tuesday October 18, 2016.

1. Let V be a finite-dimensional vector space over a field F. A linear transformation $T \in \mathrm{End}(V)$ is called *semisimple* if whenever W is an T-invariant subspace (that is $T(W) \subseteq W$) there exists a complementary T-invariant subspace W'. Here *complementary* means that $V = W \oplus W'$.

(a) If f is an irreducible polynomial in $F[x]$, let $V(f)$ be the subspace killed by a power of $f(T)$. Prove that V is the direct sum of the $V(f)$ as f runs over the irreducibles. Explain what this result has to do with the structure theory of finitely generated modules over a PID.

Remark: if F is algebraically closed, the irreducibles all have the form $f(x) = x - \lambda$ with $\lambda \in F$, and the $V(f)$ are called the *generalized eigenspaces*.

(b) Prove that T is semisimple if and only if its minimal polynomial m_T has no repeated irreducible factor.

(c) Let $n = \dim(V)$. Prove that $T^N = 0$ for some $N \geqslant 1$ if and only if $T^n = 0$. In this case T is *nilpotent*. Prove that with respect to some basis $v_1, \cdots, v_n$ the matrix of T is upper triangular, and indeed $Tv_i = v_{i-1}$ or $Tv_i = 0$.

(d) (**Jordan canonical form**) If $\lambda \in F$ the $k \times k$ *Jordan block* is the matrix

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

Use (c) to show that over an algebraically closed field T may be written as a direct sum of Jordan blocks.

(e) (**Jordan decomposition**.) The linear transformation T is *unipotent* if $T - I_V$ is nilpotent. Assume that F is algebraically closed and T is invertible. Show that T may be written *uniquely* as $T_s T_u$ where T_s and T_u commute, with T_s semisimple and T_u unipotent. **Hints:** To prove uniqueness, it will be useful to show that T_s and T_u preserve the generalized eigenspaces, and so reduce to the case T has only one eigenvalue λ; in this special case, T_s must be the scalar matrix λI. To this end you may show that if U is a transformation that commutes with T, then U preserves the generalized eigenspaces; then apply this with $U = T_s$ or T_u.

[The next two exercises are from the end of Lang, Chapter 2, page 115.]

2. Let $\mathfrak{p}$ be a prime of the commutative ring A. Let $S = A - \mathfrak{p}$. Observe that S is a multiplicative set and consider $A_\mathfrak{p} = S^{-1}A$. Show that $A_\mathfrak{p}$ has a unique maximal ideal.

3. Show that if A is a principal ideal domain and S is a multiplicative set then $S^{-1}A$ is a principal ideal domain.

4. Let $A = \mathbb{Z}[\sqrt{-5}]$ and $\mathfrak{p} = \{a + b\sqrt{-5} | a \equiv b \bmod 2\}$. Show that $\mathfrak{p}A_\mathfrak{p} = \alpha A_\mathfrak{p}$ where $\alpha = 1 + \sqrt{-5}$. Conclude that $A_\mathfrak{p}$ is a principal ideal domain with one nonzero prime ideal.