Math 210A Homework

Due Tuesday October 11, 2016.

Dedekind rings are introduced in Lang on page 88. Noetherian rings are discussed in Chapters 4 and 10. We will only need the definitions of both.

Let $R$ be a commutative ring. We recall that $R$ is called a domain if $xy = 0$ for $x, y \in R$ implies that $x = 0$ or $y = 0$. A ring $R$ is a domain if and only if it can be embedded in a field. If $F$ is a field containing $R$, then $F$ is called the field of fractions of $R$ if every element of $F$ is of the form $x/y$ with $x, y \in R$ and $y \neq 0$. The field of fractions is uniquely determined up to isomorphism. We will prove all these facts when we consider localization. But for example, the field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

If $R$ is a domain and $F$ its field of fractions, then a fractional ideal $a$ of $R$ is a nonzero, finitely-generated $R$-submodule of $F$. We can multiply fractional ideals: $ab$ is the additive subgroup of $F$ generated by elements $ab$ with $a \in a$ and $b \in b$. It is a fractional ideal.

A commutative ring $R$ is called Noetherian if every ascending chain of ideals terminates. That is, if $a_1 \subseteq a_2 \subseteq a_3 \subseteq \cdots$ are ideals, then eventually $a_n = a_{n+1} = a_{n+2} = \ldots$ for sufficiently large $n$.

1. (a) Prove that a principal ideal domain is Noetherian.

(b) Prove that $\mathbb{Z}[\sqrt{-5}]$ is Noetherian.
2. Assume that $R$ is a Noetherian domain.

(a) Prove that a nonzero $R$-submodule of $F$ is a fractional ideal if and only if $a \subseteq cR$ for some $c \in F$.

(b) The fractional ideals in $R$ form a monoid under multiplication. The ring $R$ is called a Dedekind ring if they form a group. $R$ itself will serve as the identity. Prove that a principal ideal domain is a Dedekind ring.

3. An example of an ideal in a Dedekind ring that is not principal. Let $R = \mathbb{Z}[\sqrt{-5}]$. It may be shown that $R$ is a Dedekind ring. Let

$$I = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}, a \equiv b \mod 2\}.$$ 

The exercise will show that this ideal is not principal.

(a) Let $R^* = R - \{0\}$ and $I^* = I - \{0\}$. (Some authors use $R^*$ for the multiplicative group of nonzero elements of a ring $R$, but I will use $R^\times$ for that.) Observe that if $N : R \rightarrow \mathbb{Z}$ is the map $N(z) = |z|^2$, so $N(a + b\sqrt{-5}) = a^2 + 5b^2$, then $N(zw) = N(z)N(w)$. So $N(R^*)$ and $N(I^*)$ are multiplicatively closed subsets of $\mathbb{Z}$. Show that the smallest two nonzero elements of $N(I^*)$ are 4 and 6, while the smallest nonzero elements of $N(R^*)$ are 1 and 4.

(b) Use (a) to show that $I$ is not a principal ideal.

(c) Prove that $I$ is a maximal ideal. (Hint: what is the index $[R : I]$ as abelian groups?)

(d) To prove that $R$ is a Dedekind domain we need to know that every fractional ideal is invertible. We will not prove this right now, but show that $I^2 = 2R$ and deduce that $I$ is invertible.
4. **An example of an ideal that is projective but not free.** Notations will be as in the last problem.

   (a) Prove that $I$ is not a free module.

   (b) Define $f : R \oplus R \rightarrow I \oplus I$ be multiplication by the matrix

   $$
   M = \begin{pmatrix}
   1 + \sqrt{-5} & 2 \\
   2 & 1 - \sqrt{-5}
   \end{pmatrix}.
   $$

   Show that $f$ is an isomorphism $R \oplus R \rightarrow I \oplus I$. Conclude that $I$ is a projective module.