Math 210A Homework

Due Tuesday October 11, 2016.

Dedekind rings are introduced in Lang on page 88. Noetherian rings are discussed in Chapters 4 and 10. We will only need the definitions of both.

Let R be a commutative ring. We recall that R is called a *domain* if $xy = 0$ for $x, y \in R$ implies that $x = 0$ or $y = 0$. A ring R is a domain if and only if it can be embedded in a field. If F is a field containing R, then F is called the *field of fractions* of R if every element of F is of the form x/y with $x, y \in R$ and $y \neq 0$. The field of fractions is uniquely determined up to isomorphism. We will prove all these facts when we consider localization. But for example, the field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

If R is a domain and F its field of fractions, then a *fractional ideal* $\mathfrak{a}$ of R is a nonzero, finitely-generated R-submodule of F. We can multiply fractional ideals: $\mathfrak{ab}$ is the additive subgroup of F generated by elements ab with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. It is a fractional ideal.

A commutative ring R is called *Noetherian* if every ascending chain of ideals terminates. That is, if $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$ are ideals, then eventually $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \mathfrak{a}_{n+2} = \ldots$ for sufficiently large n.

1. (a) Prove that a principal ideal domain is Noetherian.

(b) Prove that $\mathbb{Z}[\sqrt{-5}]$ is Noetherian.

2. Assume that R is a Noetherian domain.

(a) Prove that a nonzero R-submodule of F is a fractional ideal if and only if $\mathfrak{a} \subseteq cR$ for some $c \in F$.

(b) The fractional ideals in R form a monoid under multiplication. The ring R is called a *Dedekind ring* if they form a group. R itself will serve as the identity. Prove that a principal ideal domain is a Dedekind ring.

3. **An example of an ideal in a Dedekind ring that is not principal.** Let $R = \mathbb{Z}[\sqrt{-5}]$. It may be shown that R is a Dedekind ring. Let

$$I = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}, a \equiv b \bmod 2\}.$$

The exercise will show that this ideal is not principal.

(a) Let $R^* = R - \{0\}$ and $I^* = I - \{0\}$. (Some authors use R^* for the multiplicative group of nonzero elements of a ring R, but I will use $R^\times$ for that.) Observe that if $N : R \longrightarrow \mathbb{Z}$ is the map $N(z) = |z|^2$, so $N(a + b\sqrt{-5}) = a^2 + 5b^2$, then $N(zw) = N(z)N(w)$. So $N(R^*)$ and $N(I^*)$ are multiplicatively closed subsets of $\mathbb{Z}$. Show that the smallest two nonzero elements of $N(I^*)$ are 4 and 6, while the smallest nonzero elements of $N(R^*)$ are 1 and 4.

(b) Use (a) to show that I is not a principal ideal.

(c) Prove that I is a maximal ideal. (Hint: what is the index $[R : I]$ as abelian groups?)

(d) To prove that R is a Dedekind domain we need to know that every fractional ideal is invertible. We will not prove this right now, but show that $I^2 = 2R$ and deduce that I is invertible.

4. **An example of an ideal that is projective but not free**. Notations will be as in the last problem.

(a) Prove that I is not a free module.

(b) Define $f : R \oplus R \longrightarrow I \oplus I$ be multiplication by the matrix

$$M = \begin{pmatrix} 1+\sqrt{-5} & 2 \\ 2 & 1-\sqrt{-5} \end{pmatrix}.$$

Show that f is an isomorphism $R \oplus R \longrightarrow I \oplus I$. Conclude that I is a projective module.