Math 210A Homework 1

Due Tuesday October 4, 2016.

Problem 1. Let R be a ring, $\{M_i\}_{i\in I}$ an indexed family of R-modules. Define the *direct product* $\prod_i M_i$ to be the Cartesian product, that is, the set of families $(m_i)_{i\in I}$ with $m_i \in M_i$. This is itself an R-module with the multiplication and addition componentwise. The *direct sum* $\bigoplus_i M_i$ is defined to be the submodule in which $m_i = 0$ for all but finitely many i. Thus $\prod_i M_i = \bigoplus_i M_i$ if the index set I is finite, or more generally if all but finitely many M_i are zero; but not otherwise.

(a) Prove the *universal property of the direct product*: denote $P = \prod_i M_i$. If $j \in I$ define a homomorphism $p_j : P \longrightarrow M_j$ by having p_j send $(m_i)_{i\in I}$ to m_j. The maps p_j are called the *projections* onto the components of the product. The universal property asserts that given a module J and a family of homomorphisms $f_i : J \longrightarrow M_i$ there is a *unique* homomorphism $f : J \longrightarrow P$ such that $f_i = p_i \circ f$.

(b) Prove that the universal property of the product characterizes P up to isomorphism: that is, if P' is another module with homomorphisms $p'_i : P' \longrightarrow M_i$ having the same universal property, show that P and P' are isomorphic. The best solution to this does not involve the definition of P but only the universal property.

(c) Define homorphisms $\rho_i : M_i \longrightarrow \bigoplus_{i\in I} M_i$ by letting $\rho_j(m_j)$ be the family $(n_i)_{i\in I}$ where $n_i = m_i$ if $i = j$, 0 otherwise. Use these to formulate a universal property for the direct sum similar to (a).

Problem 2. If $|I| = 2$ and $\{M_i\}$ consists of the two modules M and N we will use the notation $M \times N$ for the direct product, or $M \oplus N$ for the direct sum. They are the same object, since $|I|$ is finite, but the notation $M \times N$ is appropriate if we are emphasizing the maps p_i, which we will now denote $p : M \times N \longrightarrow M$ and $q : M \times N \longrightarrow N$. Similarly the maps ρ_i will be denoted $\rho : M \longrightarrow M \oplus N$ and $\sigma : N \longrightarrow M \otimes N$. Let $P = M \times N = M \oplus N$. We can use the four maps p, q, ρ, σ together as follows.

(a) Prove that $p \circ \rho = 1_M$, $p \circ \sigma = 0$ (the zero map $N \longrightarrow M$) and $q \circ \sigma = 1_N$, $q \circ \rho = 0$, and that $\rho \circ p + \sigma \circ q = 1_P$ in the sense that if $x \in P$ then $x = \rho(p(x)) + \sigma(q(x))$.

(b) Show that this property characterizes $M \times N$ up to isomorphism: that is, given a module P with four maps $p : P \longrightarrow M$, $q : P \longrightarrow N$, $\rho : M \longrightarrow P$ and $\sigma : N \longrightarrow P$ satisfying the properties in (a), then $P \cong M \times N$.

(c) Let M, N be R-modules. Let $\mathrm{Hom}_R(M, N)$ be the abelian group of maps $f : M \longrightarrow N$ such that $f(0) = 0$, $f(x + y) = f(x) + f(y)$ and $f(rx) = rf(x)$ for $r \in R$, $x \in M$. If R is commutative, show that $\mathrm{Hom}_R(M, N)$ is an R-module with $(rf)(x) = r \cdot f(x) = f(rx)$.

(d) Use (b) to prove that $\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N)$ and $\mathrm{Hom}_R(M, N_1 \oplus N_2) \cong \mathrm{Hom}_R(M, N_1) \oplus \mathrm{Hom}_R(M, N_2)$.