MATH 122. MAPS WITH TENSOR PRODUCTS

In this handout, we work out some examples of isomorphisms involving tensor products of vector spaces. The three basic principles are: (i) to *construct* maps involving tensor product spaces we should never use bases and should instead let suitable "bilinearity" of formulas do all of the work, (ii) to prove *properties* of maps among tensor product spaces we may have to use bases (though not always; this is a matter of experience), and (iii) when checking identities among different ways of constructing linear maps, or proving that an abstract "tensorial" construction recovers some concrete construction (that also depends *linearly* on the input), it suffices to check by chasing elementary tensors.

Point (iii) merits further emphasis. In principle, when verifying an identity among *linear* maps between tensor product spaces one can chase what happens on an arbitrary linear combination of elementary tensors; however, this often makes things a big mess, and so one should just chase elementary tensors (which suffice: they do span the tensor product spaces, after all). Also, although one may note that in the spirit of cutting down one's work it is enough to even just chase what happens on tensor products of vectors from chosen bases, by treating *all* elementary tensors in a uniform way one usually gets cleaner calculations without a profusion of subscripts. When in doubt, compute with elementary tensors from bases; experience will let you recognize when you can avoid it.

1. THE DUAL MAP

Let V and V' be finite-dimensional vector spaces over a field F. Using the general linear isomorphism $\operatorname{Hom}(V, V') \simeq V' \otimes V^*$ and the "double duality" linear isomorphism $V' \simeq V'^{**}$ (that associates to any $v' \in V'$ the "evaluation" functional $e_{v'} : V'^* \to F$ in the double dual that sends $\ell' \in V'^*$ to $\ell'(v')$), we get a string of natural isomorphisms

$$(1) \qquad \operatorname{Hom}(V, V') \simeq V' \otimes V^* \simeq V'^{**} \otimes V^* \simeq V^* \otimes V'^{**} \simeq \operatorname{Hom}(V'^*, V^*)$$

where the third step uses the "flip" isomorphism from class and the final step is again the general Hom-tensor isomorphism (applied now to the duals of the original spaces).

Writing θ to denote this composite isomorphism, to any linear map $T : V \to V'$ we have naturally associated via (1) a linear map $\theta(T) : V'^* \to V^*$ such that $\theta(T)$ depends linearly on T (in the sense of the linear structure on Hom-spaces, of course). What could $\theta(T)$ be? It is natural to guess that it is just the dual map T^*, but this really requires a proof (and is good practice with making sure one understands what is going on).

To prove $\theta(T) = T^*$, one method is to choose bases of V and V' and to then chase with matrices through all steps above. This can be done and isn't too messy. However, a more insightful method that avoids the bases is to exploit the following observation: the *proposed* formula $\theta(T) = T^*$ has both sides that depend linearly on the input T. Thus, rather than checking it for all T, it suffices to check on a set of T's that span the source. But which T's should we use?

Note that elementary tensors $v' \otimes \ell$ span the term $V' \otimes V^*$, and hence the go over to a spanning set on the left in (1). We shall verify the result for the T's that arise in this way. Equivalently, what we shall do is this: we pick an elementary tensor $v' \otimes \ell$ in the second term and then chase it out to both ends, say getting a linear map $T : V \to V'$ and a linear map $\widetilde{T} : V'^* \to V^*$. It must be the case that $\widetilde{T} = \theta(T)$ (why?), and the T's that we have just obtained are necessarily a spanning set of $\operatorname{Hom}(V, V')$. Hence, if we can directly check $\widetilde{T} = T^*$ then we will have verified the desired identity on a spanning set of $\operatorname{Hom}(V, V')$ and hence we'll be done.

To summarize, we pick an elementary tensor "in the middle" and chase it out to both ends, and we aim to check that those outputs will be related in the desired manner. (**Warning**: This trick

of chasing *general* elementary tensors out from the middle is nifty, but it cannot always be relied upon because it requires that all intervening isomorphisms may be naturally defined in a way that points "out to the ends". If this is not the case, then one has to resort to computations in terms of choices of bases, and with a little practice this isn't so terrible as it may sound.)

Now comes the computation. Choose an elementary tensor $v' \otimes \ell \in V' \otimes V^*$, so on the left it goes to the linear map $T : V \to V'$ given by $v \mapsto \ell(v)v'$. In the other direction, it goes to $e_{v'} \otimes \ell$ in the third term, whence to $\ell \otimes e_{v'}$ in the fourth term, and so gives the map of dual spaces $V'^* \to V^*$ that sends $\ell' \in V'^*$ to $e_{v'}(\ell')\ell \in V^*$. By *definition* of $e_{v'}$, we have $e_{v'}(\ell') = \ell'(v')$. Thus, our goal is to show that the map from V'^* to V^* given by $\ell' \mapsto \ell'(v')\ell$ is the dual of the map $T : v \mapsto \ell(v)v'$ from V to V'.

That is, we want $T^*(\ell') = \ell'(v')\ell$ for any $\ell' \in V'^*$. By *definition* of dual maps, $T^*(\ell') = \ell' \circ T$. Hence, we want $\ell' \circ T = \ell'(v')\ell$ in V^*. To check this equality in the dual space to V, we simply evaluate on an arbitrary $v \in V$ and hope to get the same output: does $(\ell' \circ T)(v)$ equal $(\ell'(v')\ell)(v)$? By the definition of the linear structure on V^*, $(\ell'(v')\ell)(v) = \ell'(v') \cdot \ell(v)$. Hence, we want $(\ell' \circ T)(v) = \ell'(v') \cdot \ell(v)$. By definition of T,

$$(\ell' \circ T)(v) = \ell'(T(v)) = \ell'(\ell(v) \cdot v') = \ell(v) \cdot \ell'(v')$$

since $\ell' : V' \to F$ is linear. This finishes the proof.

2. Duality and tensor products

In class, we saw how to define a natural map

$$V_1^* \otimes V_2^* \to (V_1 \otimes V_2)^*$$

satisfying

$$\ell_1 \otimes \ell_2 \mapsto (v_1 \otimes v_2 \mapsto \ell_1(v_1)\ell_2(v_2)).$$

Recall that the construction of this required two steps: first we had to check that for any pair $\ell_i \in V_i^*$ the proposed linear functional on $V_1 \otimes V_2$ (sending elementary tensors $v_1 \otimes v_2$ to the proposed value $\ell_1(v_1)\ell_2(v_2)$) made sense – this amounts to the fact that $\ell_1(v_1)\ell_2(v_2) \in F$ depends bilinearly on the pair of vectors v_1 and v_2 when the ℓ_j's are fixed – and then we had to verify this this functional as an element of $(V_1 \otimes V_2)^*$ depends bilinearly on the pair of vectors ℓ_1 and ℓ_2 (in the respective dual vector spaces).

This latter verification amounted to certain identities among *linear* functionals on $V_1 \otimes V_2$, and to verify such identities it was sufficient to compare evaluations on members $v_1 \otimes v_2$ of the spanning set of elementary tensors in $V_1 \otimes V_2$. In such cases, the "evaluated" identities boiled down to another property of the "formula" $\ell_1(v_1)\ell_2(v_2) \in F$, namely that it also depends bilinearly on the pair of vectors ℓ_1 and ℓ_2 when the v_j's are fixed.

Roughly speaking, the construction of this natural map from the tensor product of dual spaces to the dual of the tensor product space comes down to the fact that the expression $\ell_1(v_1)\ell_2(v_2)$ depending on four quantities is linear in any one of them when all others are fixed. Having made the linear map

$$V_1^* \otimes V_2^* \to (V_1 \otimes V_2)^*$$

we want to show it is an isomorphism. In this case, we will chase bases in a simple manner. Let $\{v_{i,1}\}$ and $\{v_{j,2}\}$ be ordered bases of V_1 and V_2, so $\{v_{i,1} \otimes v_{j,2}\}$ is a basis of $V_1 \otimes V_2$ and $\{v_{i,1}^* \otimes v_{j,2}^*\}$ is a basis of $V_1^* \otimes V_2^*$. It suffices to prove that this basis of $V_1^* \otimes V_2^*$ goes over to the basis of $(V_1 \otimes V_2)^*$ that is dual to the basis $\{v_{i,1} \otimes v_{j,2}\}$ of $V_1 \otimes V_2$. Thus, it suffices to show that the functional on $V_1 \otimes V_2$ induced by $v_{i,1}^* \otimes v_{j,2}^*$ sends $v_{r,1} \otimes v_{s,2}$ to 1 if $(r, s) = (i, j)$ and to 0 otherwise. But this is clear: the evaluation of this functional on $v_{r,1} \otimes v_{s,2}$ is $v_{i,1}^*(v_{r,1})v_{j,2}^*(v_{s,2})$, and this is indeed 1 when

$r = i$ and $s = j$ and it is 0 otherwise (due to the definition of $\{v_{i,1}^*\}$ and $\{v_{j,2}^*\}$ as dual bases to $\{v_{i,1}\}$ and $\{v_{j,2}\}$ respectively).

3. Associativity isomorphism

We claim that there is a unique isomorphism

$$T : V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$$

that satisfies $v_1 \otimes (v_2 \otimes v_3) \mapsto (v_1 \otimes v_2) \otimes v_3$ (for $v_i \in V_i$). Let us first check uniqueness. In general, $V \otimes W$ is spanned by elementary tensors $v \otimes w$, but one can get away with less: if v and w merely run through *spanning sets* of V and W, then the $v \otimes w$'s will span $V \otimes W$. Indeed, this follows from two facts: any elementary tensor $(\sum a_i v_i) \otimes (\sum b_j w_j)$ in $V \otimes W$ is equal to the linear combination $\sum a_i b_j v_i \otimes w_j$ of the $v_i \otimes w_j$'s, and all elements of $V \otimes W$ are themselves finite linear combinations of elementary tensors in *arbitrary* vectors from V and W.) Thus, since the $v_2 \otimes v_3$'s span $V_2 \otimes V_3$, we conclude that the "elementary tensors" $v_1 \otimes (v_2 \otimes v_3)$ span $V_1 \otimes (V_2 \otimes V_3)$ even though in general many elements of $V_2 \otimes V_3$ are *not* of the form $v_2 \otimes v_3$. In view of this special spanning set for $V_1 \otimes (V_2 \otimes V_3)$, the proposed *linear* map T is certainly unique if it exists. Why does T exist?

The basic principle is that the proposed "formula" $(v_1 \otimes v_2) \otimes v_3$ in the vector space $(V_1 \otimes V_2) \otimes V_3$ is linear in each of the v_i's when the others are fixed. This, as we shall see, is why T exists as a well-defined linear map on $V_1 \otimes (V_2 \otimes V_3)$. To be precise, the definition of T simply requires us to construct a bilinear pairing

$$B : V_1 \times (V_2 \otimes V_3) \to (V_1 \otimes V_2) \otimes V_3$$

that satisfies $(v_1, v_2 \otimes v_3) \mapsto (v_1 \otimes v_2) \otimes v_3$. This formula is certainly not a *definition* for B, as we are merely specifying its value on certain special pairs. In this case, the trick to handling the well-definedness issues is to treat the variables of B separately in a suitable order. (Much more elaborate examples will be given in a later handout.)

For the present case, the idea is to fix v_1 and view the problem as that of making the *linear* map $B(v_1, \cdot)$ from $V_2 \otimes V_3$ to $(V_1 \otimes V_2) \otimes V_3$. That is, we fix v_1 and aim to make a linear map

$$B_{v_1} : V_2 \otimes V_3 \to (V_1 \otimes V_2) \otimes V_3$$

satisfying

$$v_2 \otimes v_3 \mapsto (v_1 \otimes v_2) \otimes v_3.$$

Aha, now we are back in familiar territory: the right side is bilinear in the pair v_2 and v_3 (check!), so the linear map B_{v_1} indeed exists (and is uniquely determined). Thus, we get a pairing of sets

$$\beta : V_1 \times (V_2 \otimes V_3) \to (V_1 \otimes V_2) \otimes V_3$$

via $(v_1, t) \mapsto B_{v_1}(t)$.

Now we have to analyze β: is it bilinear in its two variables v_1 and t? For fixed v_1 the linearity in t is just the fact that B_{v_1} is rigged as a linear map. Consider the problem of linearity in v_1 for fixed t. That is, is $v_1 \mapsto B_{v_1}(t)$ a linear map? This may seem tricky, as we don't have a "general formula" for $B_{v_1}(t)$ except for t's that are elementary tensors. Fortunately, we can exploit linearity of $B_{v_1}(t)$ in t to reduce our problem to the case of linearity in v_1 only for such special t's.

To see how this goes, suppose we can express t as a linear combination $\sum c_i t_i$ in $V_2 \otimes V_3$. In this case, for any v_1 we see that $B_{v_1}(t)$ is equal to the corresponding linear combination $\sum c_i B_{v_1}(t_i)$, and so the linearity problem in a varying v_1 for the fixed vector t is easily reduced to the linearity problem in a varying v_1 for each of the t_i's separately (Check!). Thus, in this way we may reduce ourselves to the case when t ranges through a spanning set in $V_2 \otimes V_3$, and so it suffices to consider t's that are elementary tensors. That is, it suffices to prove that $B_{v_1}(v_2 \otimes v_3)$ depends linearly on

v_1 for *fixed* $v_2 \in V_2$ and $v_3 \in V_3$. But this special value is $(v_1 \otimes v_2) \otimes v_3$, which visibly does have linear dependence on v_1 when v_2 and v_3 are fixed!

Remark 3.1. The preceding example is a bit special in the sense that in the tensor product $V_1 \otimes (V_2 \otimes V_3)$ at least one of the two factors (namely V_1) involves no tensors. A more sophisticated problem is to prove the existence and uniqueness of a linear isomorphism

$$T : (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \simeq V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)$$

satisfying $(v_1 \otimes v_2) \otimes (v_3 \otimes v_4) \mapsto v_1 \otimes ((v_2 \otimes v_3) \otimes v_4)$ for $v_i \in V_i$. Once one sees how to make such a T, then one can figure out the zillion other variants on such a map that may leap to mind (as we permit more V_i's and other ways of setting the parentheses).

The essential distinction is that whereas before we could translate the problem into that of associating a linear map to a member of some V_i, now the problem is that of associating a linear map $V_2 \otimes V_3 \to V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)$ to an elementary tensor $v_1 \otimes v_2 \in V_1 \otimes V_2$; this creates yet a new layer of difficulty (or so it may seem) until we see the trick: we really should fix the *ordered pair* $(v_1, v_2) \in V_1 \times V_2$ and try to map a linear map

$$B_{v_1,v_2} : V_3 \otimes V_4 \to V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)$$

satisfying $v_3 \otimes v_4 \mapsto v_1 \otimes ((v_2 \otimes v_3) \otimes v_4)$ (to be done by the usual bilinearity considerations in v_3 and v_4, with v_1 and v_2 fixed) such that the resulting pairing of sets

$$V_1 \times V_2 \to \mathrm{Hom}(V_3 \otimes V_4, V_1 \otimes ((V_2 \otimes V_3) \otimes V_4))$$

given by $(v_1, v_2) \mapsto B_{v_1,v_2}$ is *itself* bilinear, and so on.

This general problem of pairings of higher-order tensors can be addressed in a systematic manner that takes care of all such problems at once (though we don't discuss it here). Just keep in mind that the preceding worked example of a 3-fold tensor pairing is slightly more special than what one has to confront in more general situations.

With diligence and practice, these sorts of arguments really will become mechanical. Always keep in mind the general principle of "multilinearity" provides the engine that makes it all work; for example, the preceding argument used that $(v_1 \otimes v_2) \otimes v_3$ is linear in *each* of the v_i's when all others are held fixed. This is the property that always predicts when (and is used to prove that) various tensorial constructions are well-posed.

4. Trace pairings

We now combine everything: the dual-tensor, Hom-tensor, and "flip" isomorphisms. For finite-dimensional V and V', we get an isomorphism

$$\mathrm{Hom}(V, V')^* \simeq (V' \otimes V^*)^* \simeq V'^* \otimes V^{**} \simeq V'^* \otimes V \simeq V \otimes V'^* \simeq \mathrm{Hom}(V', V).$$

This is interesting: we have naturally identified $\mathrm{Hom}(V, V')$ and $\mathrm{Hom}(V', V)$ as dual to each other. That is, if L denotes this composite isomorphism, we have constructed a non-degenerate bilinear form

$$B = B_{V,V'} : \mathrm{Hom}(V', V) \times \mathrm{Hom}(V, V') \to F$$

via $B(T', T) = (L^{-1}(T'))(T) \in F$. In other words, given two linear maps $T' : V' \to V$ and $T : V \to V'$ we have provided a recipe to construct an element $B(T', T) \in F$ in a manner that depends *bilinearly* on the pair T and T'. What could this number be?

We know a couple of ways of extracting numbers from linear maps, such as traces and determinants, but these only apply to self-maps of vector spaces. Thus, for example, the self-maps $T' \circ T : V \to V$ and $T \circ T' : V' \to V'$ have traces and determinants. A moment's reflection

(check!) shows that $\text{tr}_V(T' \circ T)$ and $\text{tr}_{V'}(T \circ T')$ do depend bilinearly on the pair T and T' (due to the linearity of trace in its argument), whereas such bilinearity fails for the determinant analogues (since determinant does not have good interaction with linear operations in self-maps).

Thus, we are led to guess that perhaps $B(T', T)$ is either $\text{tr}_V(T' \circ T)$ or $\text{tr}_V(T \circ T')$. But which one? Fortunately, these two traces are the same! Indeed, the formation of the trace of a square matrix is insensitive to switching the order of multiplication when it is applied to a product of square matrices of the same size, and the same argument works in general: for any $n \times n'$ matrix (a_{ij}) and any $n' \times n$ matrix (b_{rs}), the products in both orders are $n \times n$ and $n' \times n'$ matrices whose respective traces can be directly computed to be the same: $\sum_i \sum_j a_{ij} b_{ji} = \sum_r \sum_s b_{rs} a_{sr}$.

(A closer inspection of the preceding tensorial construction of B does permit one to deduce by pure thought that $B_{V,V'}(T', T) = B_{V',V}(T, T')$, and so verifying the desired trace identity as we shall do below does permit one to recover the "product invariance" of traces that we just checked by hand using matrices, but we leave contemplation of this issue to the interested reader.)

Let us now verify that $B(T', T)$ is equal to the common trace of $T' \circ T$ and $T \circ T'$. This can certainly be verified directly by picking bases of V and V' and computing everything in terms of matrices relative to these bases and their dual bases (and elementary tensor products thereof). We encourage the reader to carry out such a calculation, and in what follows we will show an alternative method (that may seem a bit too sneaky or complicated or long, but shows that it *is* possible to pull off the proof with virtually no use of bases, if that may appeal to you). Once again we'll use the principle of chasing elementary tensors out from the middle of a string of isomorphisms.

Consider the proposed identity $B(T', T) = \text{tr}_V(T' \circ T)$. In terms of the "left pairing" linear isomorphisms $\text{Hom}(V', V) \simeq \text{Hom}(V, V')^*$ given by $T' \mapsto B(T', \cdot)$ and $T' \mapsto \text{tr}_V(T' \circ (\cdot))$, we want to prove that these linear maps agree for all T'. Rather than check for all T', we may use the evident *linearity* in T' for both formulas to reduce to checking for T' in a (well-chosen) spanning set of $\text{Hom}(V', V)$. But which spanning set should we use?

Well, we pick an elementary tensor $\ell' \otimes v$ in $V'^* \otimes V$ and we chase it out to both ends of the long string of isomorphisms: such tensors give rise to elements $T' \in \text{Hom}(V', V)$, and the T' that arise in this way are certainly a spanning set of $\text{Hom}(V', V)$ (why?). Hence, we will check the result for each such T': we will compute T' in terms of ℓ' and v, and we will also compute the linear functional that we get on $\text{Hom}(V, V')$ from $\ell' \otimes v$. We then will check that the functional we obtain is exactly $\text{tr}_V(T' \circ (\cdot))$.

Note that to compare two *linear* functionals on $\text{Hom}(V, V')$, we do not need to compare values on all $T \in \text{Hom}(V, V')$, but rather just those from a spanning set: a convenient spanning set is of course the set of T's of the form $\ell(\cdot)v'$ for $\ell \in V^*$ and $v' \in V'$ (i.e., those arising from elementary tensors in $V' \otimes V^*$).

Now we do the computation. Going to the right from $\ell' \otimes v$, in $\text{Hom}(V', V)$ we get the linear map $T' : v' \mapsto \ell'(v')v$. Going to the left, we get $\ell' \otimes e_v$ in $V'^* \otimes V^{**}$ (where e_v is the "evaluate on v" linear functional on V^*), and hence in $(V' \otimes V^*)^*$ we get the functional $v' \otimes \ell \mapsto e_v(\ell)\ell'(v') = \ell(v)\ell'(v')$. Thus, the functional we get on $\text{Hom}(V, V')^*$ sends a linear map $T : V \to V'$ of the form $\ell(\cdot)v'$ (i.e., a T that come from an elementary tensor $v' \otimes \ell \in V' \otimes V^*$) to $\ell(v)\ell'(v')$.

Our problem is therefore reduced to this: prove that the trace of the composite (the order of composition doesn't matter!) of the linear maps $\ell(\cdot)v'$ (from V to V') and $\ell'(\cdot)v$ (from V' to V) is equal to $\ell(v)\ell'(v')$. The composite $(\ell(\cdot)v') \circ (\ell'(\cdot)v)$ from V' to V' sends $v'_1 \in V'$ to $(\ell(\cdot)v')(\ell'(v'_1)v) = \ell(\ell'(v'_1)v)v' = \ell'(v'_1)\ell(v)v' = \ell(v)\ell'(v'_1)v'$. In other words, this is the map $V' \to V'$ that projects onto the line spanned by v' with multiplier coefficient function $\ell(v)\ell'$. The trace of this self-map must be proved to equal $\ell(v)\ell'(v')$. The scalar $\ell(v)$ passes through the trace, so we can ignore it: it suffices to prove that the self-map $\ell'(\cdot)v'$ from V' to V' has trace $\ell'(v')$.

The case $v' = 0$ is trivial, and otherwise we may consider a basis of V' containing v' as the first basis vector. With respect to this basis, the map sends the first basis vector v' to $\ell'(v')v'$ and sends all other basis vectors to multiples of the first basis vector v'. Hence, the matrix for the map with respect to this basis has upper left entry $\ell'(v')$ and has all other diagonal entries equal to 0. Thus, the trace is $\ell'(v')$ as desired.