Math 122: Project on induced representations

You may discuss the project with myself, Matt Larson, or with one another. You may *not* show another student your written work.

You may use anything you find in Dummit and Foote, Lang's *Algebra* or anything you find posted on the course web page. You may even find some of the questions proved verbatim.

By a *G-module V* we mean a $\mathbb{C}[G]$-module. Thus G-modules are equivalent to representations of G. In contrast with notation used in Lecture 8, here V^G does *not* mean the module of invariants. It is the induced module.

1. Let G be a finite group and let V, W be $\mathbb{C}[G]$ modules, not necessarily irreducible. Let χ_V and χ_W be the characters. Explain why

$$\dim\ \mathrm{Hom}_{\mathbb{C}[G]}(V, W) = \langle \chi_V, \chi_W \rangle.$$

2. Let H be a subgroup of the finite group G. Let V be an H-module. Prove that there exists a G-module V^G, unique up to isomorphism, such that for every G-module

$$\dim\ \mathrm{Hom}_{\mathbb{C}[G]}(V^G, W) = \dim\ \mathrm{Hom}_{\mathbb{C}[H]}(V, W).$$

Hint: Explain why it is sufficient to check this for W irreducible. Let $V_1, \cdots, V_h$ be the distinct irreducible G-modules. Postulate a solution of the form

$$V^G = \bigoplus d_i V_i.$$

What should the multiplicities d_i be?

Problem 2 suggests that the *induced representation* V^G should be a functor from the category of H-modules to the category of G-modules.

3. Let V be an H-module. Let $\pi : H \longrightarrow \mathrm{GL}(V)$ be the corresponding representation, so $\pi(h)v = hv$. Let $I(V) = I_H^G(V)$ be the space of all maps $f : G \longrightarrow V$ that satisfy

$$f(hg) = \pi(h)\big(f(g)\big), \qquad h \in H.$$

Define a map $\Pi : G \longrightarrow \mathrm{GL}(I(V))$ by

$$\big(\Pi(g)f\big)(x) = f(xg), \qquad g, x \in G.$$

(a) Prove that $\Pi(g)f \in I(V)$ if $f \in I(V)$ and $g \in G$.

(b) Prove that Π is a representation of G.

(c) If $v \in V$ define a map $\hat{v} : G \longrightarrow V$ by

$$\hat{v}(g) = \begin{cases} gv & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\hat{v} \in I(V)$ and that the map $\alpha : V \longrightarrow I(V)$ defined by $\alpha(v) = \hat{v}$ is an H-module homomorphism.

(d) Prove the following *universal property*: if $\theta : V \longrightarrow W$ is any H-module homomorphism into a G-module W then there is a unique G-module homomorphism $\Theta : I(V) \longrightarrow W$ such that $\theta = \Theta\alpha$.

(e) Prove that $I(V) \cong V^G$ as a G-module.

4. Let χ be the character of the H-module V and let $\dot{\chi} : G \longrightarrow \mathbb{C}$ be the function

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the character χ^G of the induced representation $V^G = I(V)$ is given by the formula

$$\chi^G(x) = \frac{1}{|H|} \sum_{g \in G} \dot{\chi}(gxg^{-1}).$$

5. If f is a class function on G that can be written as a difference of two characters then f is called a *virtual character* or a *generalized character*.

(a) Explain briefly why the generalized characters form a ring, called the *character ring*.

(b) Suppose that f is a generalized character and

$$\langle f, f \rangle = 1, \qquad f(1) > 0.$$

Prove that f is the character of an irreducible representation.