Lecture 9. Permutation Representations

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Permutation representations

Let $G$ be a finite group, and $X$ a set with a group action $G \times X \rightarrow X$. Let $V$ be the free complex vector space on $X$. Then $G$ acts on $V$ by linear transformations, extending the given action on the basis $X$. Thus we have a representation $\pi_X : G \rightarrow \text{GL}(V)$.

For example, let $G = S_3$ with its usual action on $X = \{1, 2, 3\}$ by permutations. With respect to this basis, $\pi_X(g)$ becomes a matrix:

\[
\pi_X((12)) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \pi_X((123)) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]
The character of the permutation representations

Diagonal entries in the matrix of $\pi_X(g)$ with respect to the basis $X$ of $V$ are then fixed points: that is, there is 1 in the $x$-th diagonal if $gx = x$, that is, if $x$ is a fixed point of $g$. From this, we see that the character $\chi_X(g)$ is the number of fixed points of $g$:

$$\chi_X(g) = |\{x \in X | gx = x\}| .$$

For example, if $G = S_3$ and $X = \{1, 2, 3\}$:

<table>
<thead>
<tr>
<th>$g$</th>
<th>1</th>
<th>(123)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_X(g)$</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We will call $\pi_X$ and $\chi_X$ the permutation representation and permutation character associated with the group action on $X$. 
The reduced character

The permutation representation $\pi_X$ is reducible (unless $|X| = 1$). Indeed, it contains a copy of the trivial representation, namely the span of the basis element $\xi = \sum_{x \in G} x$. Another submodule (which may or may not be irreducible) consist of the elements

$$V^\circ = \left\{ \sum_{x \in X} a_x x \mid \sum_{x \in X} a_x = 0 \right\}.$$

Let $\pi_X^\circ$ be the representation of $G$ on $V^\circ$. The dimension of $V^\circ$ is $|X| - 1$ and

$$V = \mathbb{C} \xi \oplus V^\circ.$$

The character $\chi_X^\circ$ of the $G$-module $V^\circ$ equals the number of fixed points minus 1, and this character may or may not be irreducible.
Example

For $S_3$, $\chi^\circ_X(g)$ is the number of fixed points, minus one:

<table>
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<tr>
<th>$g$</th>
<th>1</th>
<th>(123)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^\circ_X(g)$</td>
<td>2</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>

We will call $\pi^\circ_X$ and $\chi^\circ_X$ the reduced permutation representation and character respectively.

In this example, the reduced character is irreducible.

The reduced character may or may not be irreducible.
Irreducibility of the reduced permutation character

Let us consider whether $\pi_X^O$ is an irreducible representation or not. First, if the action on $X$ is not transitive, it can never be irreducible. This is because if there are more than one orbit:

$$X = X_1 \cup X_2 \cup \cdots \cup X_h \quad \text{(disjoint)}$$

then for each orbit $X_i$ the vector $\xi_i$ that is the sum of the elements in $X_i$ is an invariant vector, so $\pi_X^O$ will contain $h$ copies of the trivial representation.

But even if the action on $X$ is transitive, the reduced permutation representation may or may not be irreducible. We will look at couple of examples.
A test for irreducibility

First let us describe an easy way of checking whether a representation is irreducible.

**Proposition**

Let $\chi$ be the character of a representation $\pi$. Then $\pi$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

By Schur orthogonality, if $\chi$ is irreducible, then $\langle \chi, \chi \rangle = 1$. So prove the converse, we may decompose $\pi$ into distinct irreducible representations:

$$\pi \cong \bigoplus_i n_i \pi_i.$$  

Here $\pi_i$ are the distinct irreducibles.
Proof, continued

The character is:

\[ \chi = \sum_i n_i \chi_i. \]

Now using orthogonality

\[ \langle \chi, \chi \rangle = \sum_i n_i^2. \]

So if \( \langle \chi, \chi \rangle = 1 \) then \( \sum n_i^2 = 1 \), which implies that exactly one of the \( n_i \) is \( > 0 \), and that \( n_i = 1 \). This means that \( \pi \cong \pi_i \) is irreducible.
Example: $S_4$

With this in hand, we consider a couple of permutation representations. First, there is the action of $S_4$ on $X = \{1, 2, 3, 4\}$. (We considered the case of $S_3$ last week.) Here are the character values. The first row gives representatives of the conjugacy class. The second row gives the number of elements of the conjugacy class.

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<tr>
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<th>(123)</th>
<th>(12)(34)</th>
<th>(12)</th>
<th>(1234)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_X$</td>
<td>1</td>
<td>8</td>
<td>3</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$\chi_X^\circ$</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

$$\langle \chi, \chi \rangle = \frac{1}{24} (1 \cdot 3^2 + 8 \cdot 0^2 + 3 \cdot (-1)^2 + 6 \cdot 1^2 + 6 \cdot (-1)^2) = 1$$

So this character is irreducible.
Example: $D_8$

Now let us restrict this permutation representation to the dihedral group $D_8$. It is still a transitive action.

\[
\begin{array}{c|ccccc}
 & 1 & (1234) & (13)(24) & (12)(34) & (13) \\
\hline
1 & 2 \\
\hline
\chi_X & 3 & -1 & -1 & -1 & 1 \\
\hline
\end{array}
\]

\[
\langle \chi, \chi \rangle = \frac{1}{8} (1 \cdot 3^2 + 2 \cdot (-1)^2 + 1 \cdot (-1)^2 + 2 \cdot (-1)^2 + 2 \cdot 1^2) = 2.
\]

So this representation is not irreducible.
Example: $D_8$ (continued)

Still we can use this information to construct an irreducible representation of $D_8$. The derived group is the center $Z(D_8) = \langle (13)(24) \rangle$ of order 2. The quotient $D_8/Z(D_8)$ is isomorphic to $Z_2 \times Z_2$ so we find four linear characters, and we have part of the character table:

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<tr>
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<th>(13)(24)</th>
<th>(12)(34)</th>
<th>(13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^\circ_X$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

From these values, $\langle \chi^\circ_X, \chi_4 \rangle = 1$. This means we can subtract $\chi_4$ from $\chi^\circ_X$ and obtain another character which we will call $\chi_5$. 
### $D_8$ (concluded)

<table>
<thead>
<tr>
<th></th>
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<th>(13)(24)</th>
<th>(12)(34)</th>
<th>(13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi^\circ_\chi$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Question**

Can you think of another way to compute $\chi_5$ once $\chi_1, \chi_2, \chi_3, \chi_4$ have been found?
The regular representation

These two examples show that the reduced character of a transitive permutation representation may or may not be irreducible.

A particular important permutation representation is the action of $G$ on itself by left translation. The corresponding representation is called the regular representation $\pi_{\text{reg}}$. Let $\chi_{\text{reg}}$ be its character. The free vector space on $G$ is $\mathbb{C}[G]$ so we may think of the regular representation as being the representation on the $\mathbb{C}[G]$-module $\mathbb{C}[G]$. 
The character of the regular representation

**Proposition**

The character of the regular representation is

\[ \chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases} \]

To prove this, note that \( x \in G \) is a fixed point of \( g \) if and only if \( gx = x \), which means \( g = 1 \). So \( g = 1 \) has \( |G| \) fixed points, and any other element has none.
Decomposing the regular representation

Now let us decompose the regular representation into irreducibles. We recall that $d_i = \dim(V_i)$, so $d_i = \chi_i(1)$.

**Theorem**

The multiplicity of $\pi_i$ in a decomposition of $\mathbb{C}[G]$ into irreducibles is $d_i$. We have

$$\chi_{\text{reg}} = \sum_i d_i \chi_i$$

and

$$\sum d_i^2 = |G|.$$
Proof

To prove this, we note that by Maschke’s theorem, there does exist a decomposition of $\mathbb{C}[G]$ into irreducibles:

$$\mathbb{C}[G] = \bigoplus_j M_j$$

where $M_j$ are simple left ideals. If $k_i$ is the number of $M_j$ isomorphic to $\pi_i$ then we may also write

$$\mathbb{C}[G] \cong \bigoplus k_i V_i.$$  

We are claiming that the multiplicity $k_i = d_i$, the dimension of $V_i$.  

Proof, continued

Consider the character

$$\chi_{\text{reg}} = \sum_i k_i \chi_i.$$  

By Schur orthogonality, the $\chi_i$ are orthonormal, so

$$k_i = \langle \chi_{\text{reg}}, \chi_i \rangle.$$  

This may be calculated since we know the character of $\chi_{\text{reg}}$. Remember:

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise}. \end{cases}$$  

So

$$\langle \chi_{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\text{reg}}(g) \chi_i(g) = \frac{1}{|G|} |G| \chi_i(1) = \chi_i(1) = d_i.$$
Proof, concluded

We have proved that

\[ \mathbb{C}[G] = \sum_i d_i \pi_i, \quad \chi_{\text{reg}} = \sum_i d_i \chi_i. \]

Evaluating \( \chi_{\text{reg}} \) at the identity gives

\[ |G| = \chi_{\text{reg}}(1) = \sum_i d_i \chi_i(1) = \sum_i d_i^2. \]
Example

The identity $|G| = \sum_i d_i^2$ gives one way of knowing if we have found all the irreducible characters of the group, and it is useful information that can tell us in advance what the degrees of the irreducible characters will be. Let us check this for $S_3$.

For $S_3$, we quickly find three irreducible characters, namely two linear characters (the trivial and sign character) and the reduced character of the permutation representation (number of fixed points minus 1). So the $d_i$ are 1, 1, 2 and $\sum d_i^2 = 1^2 + 1^2 + 2^2 = 6$, confirming the numerology in the theorem.

Similarly for $D_8$ we found three linear characters and one of degree 2, and $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$. 

The number of irreducibles (I)

**Proposition**

Let $G$ be a finite group. Let $h$ be the number of conjugacy classes of $G$. Then $G$ has at most $h$ nonisomorphic irreducible representations.

Indeed, let $\pi_i : G \rightarrow \text{GL}(V_i)$ be nonisomorphic irreducible representations ($i = 1, 2, 3, \cdots$) and let $\chi_i$ be their characters. We saw that $\chi_i$ are elements of $L^2_{\text{class}}(G)$, which is the vector space of class functions with the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)},$$

making $L^2_{\text{class}}(G)$ a finite-dimensional Hilbert space.
The dimension of $L^2_{\text{class}}(G)$ is the number $h$ of conjugacy classes, since if $C_1, \cdots, C_h$ are the conjugacy classes their characteristic functions are obviously a basis of $L^2_{\text{class}}(G)$. On the other hand the $\chi_i$ are orthonormal by Schur orthogonality:

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

so the number of $\pi_i$ is at most $h$.

There is a loose end.

- We have proved that there are at most $h$ irreducible representations of $G$, where $h$ is the number of conjugacy classes. We will show later that actually there are exactly $h$ irreducible representations.
- We need to know that the irreducible characters span $L^2_{\text{class}}(G)$. We will prove that later.