If G acts on the set X, define $x \sim y$ if $gx = y$ for some $g \in G$.

This is an equivalence relation.

The equivalence classes are called *orbits*.

If there is only one orbit the action is called *transitive*.

In order for π_X° to be irreducible we want the action to be transitive. However this may or may not be enough.

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

We know

$$\chi_X^\circ = \sum_{i=1}^{5} n_i \chi_i$$

for some n_i. And we know $\chi_1, \chi_2, \chi_3, \chi_4$ so we can compute

$$n_i = \langle \chi_X^\circ, \chi_i \rangle$$

i	1	2	3	4	5
n_i	0	0	0	1	?

So we can subtract χ_4 and still get a character

$$\chi_X^\circ - \chi_4 = \sum_{i=5} n_i \chi_i.$$

Today:

$$\#\mathrm{irr} \leqslant \#\text{conjugacy classes}$$

By Schur orthogonality, χ_i are orthonormal and live in $L^2_{\text{class}}(G)$. And $\dim L^2_{\text{class}}(G) =$ #conj classes.

$$D_8 = \langle x, y | x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

$$x = (1234), \qquad y = (14)(23)$$

Let

$$X = \begin{pmatrix} \cos\left(\frac{2\pi}{4}\right) & \sin\left(\frac{2\pi}{4}\right) \\ -\sin\left(\frac{2\pi}{4}\right) & \cos\left(\frac{2\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\boxed{\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}}$$

There is a representation $\pi: D_8 \longrightarrow \mathrm{GL}(2, \mathbb{R}) \subseteq \mathrm{GL}(2, \mathbb{C})$ such that

$$\pi(x) = X, \pi(Y) = y.$$

$$\pi_{\text{reg}}(12) = \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}$$

with respect to the basis $1, (12), (13), (123), (23), (132)$

In Section 18.2 of DF there is stated *Wedderburn's theorem* that a "semisimple ring with minimum condition" is a direct sum of matrix rings over division rings.

The proof is relegated to the exercises.

The discussion in Lang's *Algebra* is much better.