# Lecture 8. Schur orthogonality

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## Trace of a matrix

The trace of a square matrix is the sum of its diagonal elements, or equivalently, the sum of its eigenvalues (counted with the correct multiplicity, of course). It is invariant under conjugation: if $M$ is invertible

$$\mathrm{tr}(A) = \mathrm{tr}(MAM^{-1}).$$

If $T : V \to V$ is an endomorphism of the finite-dimensional vector space $V$, define $\mathrm{tr}(T)$ to be the trace of the matrix of $T$ with respect to any basis.

If $T_1 : V_1 \to V_1$ and $T_2 : V_2 \to V_2$ are endomorphisms, we have endomorphisms $T_1 \oplus T_2$ and $T_1 \otimes T_2$ of $V_1 \oplus V_2$ and $V_1 \otimes V_2$ and

$$tr(T_1 + T_2) = \mathrm{tr}(T_1) + \mathrm{tr}(T_2), \qquad \mathrm{tr}(T_1 \otimes T_2) = \mathrm{tr}(T_1)\ \mathrm{tr}(T_2).$$

## The character of a representation

Let $\pi : G \to \mathrm{GL}(V)$ be a representation. The character of $\pi$ is the function

$$\boxed{\chi_\pi(g) = \mathrm{tr}(\pi(g)).}$$

A function $f : G \to \mathbb{C}$ is a class function if $f(gxg^{-1}) = f(x)$, so that it is constant on conjugacy classes.

Since

$$\chi_\pi(ghg^{-1}) = \mathrm{tr}(\pi(g)\pi(h)\pi(g)^{-1}) = \mathrm{tr}(\pi(h)) = \chi_\pi(h),$$

the character is a class function.

## Linear characters (I)

If $\chi : G \to \mathbb{C}^\times$ is a homomorphism, then we may identify $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$ and we obtain a one-dimensional representation. Since the trace of a $1 \times 1$ matrix is just its unique entry, the character of this one-dimensional representation is $\chi$.

We will call the character of a one-dimensional representation a linear character, and we see that these are just homomorphisms $G \to \mathbb{C}^\times$.

Every group $G$ has a linear character, namely the character of the trivial representation $1_G : G \to GL(1, \mathbb{C})$ that maps every element of $G$ to the identity. The character of the trivial representation is the function that has value $1$ on every conjugacy class.

## Linear characters (II)

If $G$ is a finite group, there is a smallest normal subgroup $N$ such that $G/N$ is abelian. Indeed, it is easy to see that $G/N$ is abelian if and only if $N$ contains all commutators $[x, y] = xyx^{-1}y^{-1}$. Note that

$$g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}].$$

This identity implies that the group $G'$ generated by the commutators is normal. This is the derived group or commutator subgroup.

We see that a quotient $G/N$ is abelian if and only if $G' \subseteq N$. Every linear character, being a homomorphism to an abelian group, has a kernel that contains all the commutators. Hence it factors through $G/G'$.

## Irreducible characters

If $\pi$ is a representation (whether irreducible or not) we will just refer to $\chi_\pi$ as a character. If $\pi$ is an irreducible representation we will call $\chi_\pi$ an irreducible character. Eventually we will prove that there are only a finite number of irreducible characters. Since every representation can be decomposed as the direct sum of irreducible representations, every character is a sum of irreducible characters.

## The character ring

**Proposition**

*Let $\chi_1$ and $\chi_2$ be characters. Then $\chi_1 + \chi_2$ and $\chi_1\chi_2$ are characters.*

Indeed, they are the characters of $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ acting on $V_1 \oplus V_2$ and $V_1 \otimes V_2$.

Let us call a class function a virtual character or a generalized character if it is the difference of two characters. Since the set of characters is closed under addition and multiplication, the virtual characters form a ring, called the character ring.

## The contragredient or dual representation

If $\pi : G \longrightarrow \mathrm{GL}(V)$ is a representation, we also have a representation on the dual space $V^*$. We define $\hat{\pi} : G \longrightarrow \mathrm{GL}(V^*)$ by $\hat{\pi}(g) = \pi(g^{-1})^*$. The use of $g^{-1}$ here is needed so that

$$\hat{\pi}(g \circ h) = \hat{\pi}(g) \circ \hat{\pi}(h).$$

### Proposition

*The character of $\hat{\pi}$ is the complex conjugate of $\chi_\pi$.*

This is because if $\varepsilon_1, \cdots, \varepsilon_d$ are the eigenvalues of $\pi(g)$, then $\varepsilon_1^{-1}, \cdots, \varepsilon_d^{-1}$ are the eigenvalues of $\pi(g^{-1})$ and of $\pi(g^{-1})^*$; but these are the conjugates of $\varepsilon_1, \cdots, \varepsilon_d$ because they are roots of unity. Taking traces,

$$\chi_{\hat{\pi}}(g) = \sum \varepsilon_i^{-1} = \overline{\chi_\pi}(g).$$

## Schur's Lemma part I

Let $R$ be a ring. By an $R$-module we will mean a left module, unless otherwise stated. Recall that a simple module $M$ is one that has no submodules except $M$ itself and $0$.

**Proposition (Schur's Lemma, Part I)**

*Let $M$ and $N$ be simple $R$-modules. If $\phi : M \longrightarrow N$ is an $R$-module homomorphism, either $\phi = 0$ or $\phi$ is an isomorphism.*

To prove this, note that $\ker(\phi)$ is a submodule of $M$, which is not $M$ if $\phi \neq 0$. Since $M$ is simple, $\ker(\phi) = 0$ proving $\phi$ is injective. Similarly $\mathrm{im}(\phi)$ is a submodule of $N$, which cannot be $0$ since $\phi \neq 0$; since $N$ is simple, $\mathrm{im}(\phi) = N$, proving that $\phi$ is surjective. Thus $\phi$ is an isomorphism.

## Schur's Lemma, Part II

**Proposition (Schur's Lemma, Part II)**

*Suppose that $R$ is an algebra over an algebraically closed field $F$ and that $M$ is a simple $R$-module that is finite-dimensional as an $F$-vector space. If $\phi \in \mathrm{End}_R(M)$ then there exists a scalar $\lambda \in F$ such that $\phi(x) = \lambda x$ for all $x \in M$.*

To prove this, since $M$ is a finite-dimensional vector space over an algebraically closed field $F$, the linear transformation $\phi$ has an eigenvalue $\lambda \in F$. Now consider the transformation $\phi - \lambda I_M$. This transformation is not injective, so by the first part of Schur's Lemma, it is the zero map.

## Hilbert space

Let $L^2(G)$ we the space of functions on $G$. It is a Hilbert space with inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

The subspace $L^2_{\text{class}}(G)$ of class functions on $G$ inherits this inner product. If $\pi$ is a representation of $G$, then $\chi_\pi \in L^2_{\text{class}}(G)$. Today we will prove:

**Theorem**

*The characters of the irreducible representations are an orthonormal basis of $L^2_{\text{class}}(G)$.*

## An analogy

We observe the relationship between this formula and Schur's Lemma, which we have already proved. Note the similarity:

$$\dim\ \mathrm{Hom}_{\mathbb{C}[G]}(V_1, V_2) = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2, \\ 0 & \text{otherwise.} \end{cases}$$

Here $V_1$ and $V_2$ are the $\mathbb{C}[G]$-modules corresponding to $\pi_1$ and $\pi_2$.

Compare this to the assertion of Schur orthogonality:

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2, \\ 0 & \text{otherwise.} \end{cases},$$

where $\chi_i$ are the characters corresponding to the modules $V_i$. We will use this parallel to prove Schur orthogonality.

## A special case

As a first step:

**Proposition**

*Let* $\pi : G \longrightarrow \mathrm{GL}(V)$ *be an irreducible representation with character* $\chi$*. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) = \begin{cases} 1 & \text{if } \pi \text{ is the trivial representation} \\ 0 & \text{otherwise.} \end{cases}$$

If $\pi$ is trivial then $\chi(g) = 1$ so $\frac{1}{|G|}\sum\chi(g) = 1$. Thus we have to show that if

$$\sum_{g\in G}\chi(g) \neq 0$$

then $\pi$ is trivial.

## Proof

Choose a basis $v_i$ of $V$ and let $v_i^*$ be the dual basis of $V^*$. Then remembering the formula

$$\mathrm{tr}(T) = \sum_i v_i^*(Tv_i)$$

we have

$$\sum_{g\in G} \chi(g) = \sum_i \sum_{g\in G} v_i^*(\pi_i(g)v_i) = \sum_i v_i^* \left( \sum_{g\in G} \pi_i(g)v_i \right).$$

This must be nonzero for some $i$, so

$$\sum_{g\in G} \pi_i(g)v_i \neq 0.$$

Let $\xi$ be this vector. Clearly $\pi(g)\xi = \xi$ for all $g$, so $\mathbb{C}\xi$ is a nonzero submodule of $V$; since $\pi$ is irreducible, $V = \mathbb{C}\xi$. Then $\pi(g)\xi = \xi$ implies that $\pi$ is the trivial representation.

## The module of invariants

Let $W$ be a $G$-module, which we do not assume to be irreducible. We define $W^G$ to be the module of invariants in $W$, defined by:

$$W^G = \{g \in G | gw = w \text{ for all } g \in G\}.$$

For example, let us consider an irreducible representation. Then the submodule of invariants is $W$ if $W$ is the trivial representation, but $0$ for any other irreducible. This is clear from the definitions.

## The dimension of the module of invariants

### Proposition

*Let $\chi$ be the character of $W$. The dimension of $W^G$ is*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g).$$

Use Maschke's theorem to decompose $W$ into irreducible, and we separate out the summands that are trivial modules:

$$W = \left(\bigoplus_{i=1}^{d} W_i\right) \oplus \left(\bigoplus_{i=d+1}^{N} W_i\right)$$

So $W_i \cong 1$ (the trivial module) for $i = 1, \cdots, d$ and the other $W_i$ are not trivial. The first summand is the module of invariants so $d = \dim(W^G)$.

## Proof, continued

Continuing from the decomposition:

$$W = \left( \bigoplus_{i=1}^{d} W_i \right) \oplus \left( \bigoplus_{i=d+1}^{N} W_i \right)$$

$$W^G = \text{ first term } = \bigoplus_{i=1}^{d} W_i$$

If $\chi_i$ is the character of $W_i$ then

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) = \sum_{i=1}^{N} \frac{1}{|G|} \sum_{g \in G} \chi_i(g) = d.$$

## $\mathrm{Hom}(V_1, V_2)$ as a module

Since $G$ acts on $V_1$ and $V_2$, and since $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ is a functor in both $V_1$ and $V_2$, we have an action of $G$ on $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, which becomes a $\mathbb{C}[G]$-module. Let $\Pi : G \longrightarrow \mathrm{Hom}(V_1, V_2)$ be the corresponding representation. If $g \in G$ and $f : V_1 \longrightarrow V_2$ is a linear transformation, then $\Pi(g)f$ is the map $\pi_2(g) f \pi_1(g^{-1})$. Here we need the $g^{-1}$ for the action on $V_1$ because $\mathrm{Hom}(V_1, V_2)$ is contravariant in $V_1$.

## The character of $\mathrm{Hom}(V_1, V_2)$

Let us compute the character of this representation $\Pi$. We do this by noting the natural isomorphism

$$\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2.$$

Because this isomorphism is natural, the representation on $V_1^* \otimes V_2$ is equivalent to the representation on $\mathrm{Hom}(V_1, V_2)$, and we use this fact to compute the character.

We will denote by $\chi_1$ and $\chi_2$ the characters of $\pi_1$ and $\pi_2$. The character of the representation on $V_1^*$ (the contragredient representation) is $\overline{\chi}_1$ and of course the character of the representation of $V_2$ is $\chi_2$. Therefore the character on $V_1^* \otimes V_2$ is $\overline{\chi_1(g)}\chi_2(g)$, and this is also the character of the representation on $\mathrm{Hom}(V_1, V_2)$.

## Schur orthogonality

Now we can prove Schur orthogonality. We take $W = \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$. We note that from the action

$$\Pi(g)f = \pi_2(g) f \pi_1(g^{-1})$$

the module of invariants is

$$W^G = \mathrm{Hom}_{\mathbb{C}[G]}(V_1, V_2).$$

The character of $W$ is $\chi(g) = \overline{\chi_1(g)}\chi_2(g)$ and therefore

$$\dim \mathrm{Hom}_{\mathbb{C}[G]}(V_1, V_2) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)}\chi_2(g) = \langle \chi_2, \chi_1 \rangle.$$

Since this is real, it is also equal to $\langle \chi_1, \chi_2 \rangle$.