If $\pi: G \longrightarrow \mathrm{GL}(V)$ is a representation of a finite group

$$\chi_\pi(g) = \mathrm{tr}\,\pi(g)$$

Easy to work with:

Only depends on the conjugacy class of g, hence is easy to describe.

	1	(123)	(12)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

If χ_1, χ_2 are characters of irreducibles π_1, π_2

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

For any finite group G, G/G' is an abelian group. (Direct product of cyclic groups.)

It is easy to compute the characters of any cyclic group.

Characters of $Z_5 = \langle x | x^5 = 1 \rangle$: let $\zeta = \zeta_5 = e^{2\pi i/5}$

	1	x	x^2	x^3	x^4
χ_1	1	1	1	1	1
χ_2	1	ζ	ζ^2	ζ^3	ζ^4
χ_3	1	ζ^2	ζ^4	ζ	ζ^3
χ_4	1	ζ^3	ζ	ζ^4	ζ^2
χ_5	1	ζ^4	ζ^3	ζ^2	ζ

Since Z_5 is abelian, every character χ is linear (ie. a hom. $G \longrightarrow \mathbb{C}^\times$)

$$\chi(ab) = \chi(a)\chi(b), \qquad \chi(x)^5 = \chi(x^5) = \chi(1) = 1$$

so $\chi(x)$ is a 5-th root of 1. Characters (all linear) of a cyclic group are easy to compute. Same is true for direct products of cyclic groups, i.e. arbitrary abelian groups.

Set of characters of S_3 is $\{a\chi_1 + b\chi_2 + c\chi_3 | a, b, c \in \mathbb{Z}, a, b, c \geqslant 0\}$

Character ring (set of virtual characters) of S_3 is $\{a\chi_1 + b\chi_2 + c\chi_3 | a, b, c \in \mathbb{Z}\}$

$$\xi = \sum_{h \in G} \pi(h) v_i$$

$$\pi(g) = \sum_{h \in G} \pi(gh) v_i = \sum_{h \in G} \pi(h) v_i = \xi \qquad h \rightarrow g^{-1}h$$

$\mathbb{C}[G]$ modules are called

G-modules

$f \in \mathrm{Hom}(V_1, V_2)$

is invariant if and only if

$$f = \pi_2(g) f \pi_1(g)^{-1}$$

$$f\pi_1(g) = \pi_2(g) f$$

$$f(gv_1) = g f(v_2)$$

i.e. if and only if

$$f \in \mathrm{Hom}_{\mathbb{C}[G]}(V_1, V_2)$$

Character table of S_4

	1	8	3	6	6
	1	(123)	(12)(34)	(12)	(1234)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	-1	2	0	0
χ_4	3	0	-1	1	-1
χ_5	3	0	-1	-1	1

Commutator subgroup is A_4. It is normal, and we know that the normal subgroups are:

$$1, V, A_4, S_4$$

$$V = \{1, (12)(34), (13)(23), (14)(23)\}$$

G	1	V	A_4	S_4
G/N	S_4	S_3	Z_2	1

$$G/V \cong S_3$$

Consider the action of G by permutions on $\{(12)(34), (13)(23), (14)(23)\}$. This is a group action whose kernel is V, hence a homomorphism $G \longrightarrow S_3$ with kernel V. So there are $|G/G'| = |Z_2| = 2$ linear characters.

We can make use of the homomorphism $G \longrightarrow G/V \cong S_3$ and pull back the characters of this group. We'll recover χ_1 and χ_2 and one more.

	1	(123)	(12)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

There is a 3-dimensional character which is #fixed points -1. (Similarly to S_3). This gives us χ_4. To check it is irreducible it is enough to compute

$$\frac{1}{|G|}\sum \langle \chi_4, \chi_4 \rangle = \frac{1}{24}(9 + 8 \times 0 + 3 \times (-1)^2 + 6 \times 1^2 + 6 \times (-1)^2) = 1$$

Why does this imply irreducible?

If $\langle \chi, \chi \rangle = 1$ for any character write $\chi = \sum d_i \chi_i$ for irr. χ_i and

$$\langle \chi, \chi \rangle = \sum d_i^2$$

$$\sum d_i^2 = 1 \quad \Rightarrow \quad \text{all } d_i = 0 \text{ except } 1$$

To get χ_5 define $\chi_5 = \chi_2\chi_4$ and prove irreducible the same way.