

Lecture 7: Characters (I)

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History

Characters of finite **abelian** groups were studied by Dirichlet who used them to prove his theorem on primes in an arithmetic progression. Then Frobenius developed a **character** theory for finite nonabelian groups.

Frobenius did not at first use representations to define characters. It was Frobenius' student Schur who interpreted Frobenius' characters as characters of representations. Frobenius, Schur and Burnside invented the subject of group character theory between 1895 and the early part of the 20th century.

Burnside was a Professor at the Royal Naval College in London. Finite group theory was therefore a professional sideline to him, but his researches were very deep.

Trace of a matrix

If $A = (a_{ij})$ is a square matrix, let $\text{tr}(A) = \sum_i a_{ii}$. Then

$$\text{tr}(AB) = \sum_{ij} a_{ij}b_{ji}.$$

This is the **trace** of A . From this identity we deduce

$$\text{tr}(AB) = \text{tr}(BA).$$

If we apply this identity with $A = TB^{-1}$ we get

$$\text{tr}(T) = \text{tr}(BTB^{-1}),$$

so the trace is unchanged by conjugation. Any $T \in \text{Mat}_n(\mathbb{C})$ is conjugate to a triangular matrix T' with the eigenvalues λ_i on the diagonal. Since $\text{tr}(T) = \text{tr}(T')$ this proves

$$\text{tr}(T) = \sum_i \lambda_i.$$

An interpretation of the trace

We will not need the following but it is worth noting.

If V is a vector space of dimension n then $\text{Mat}_n(\mathbb{C}) \cong \text{End}(V)$ so tr is a map $\text{End}(V) \rightarrow \mathbb{C}$. We have mentioned isomorphisms

$$\text{End}(V) = \text{Hom}(V, V) \cong V^* \otimes V.$$

Thus f may be interpreted as an element of $V^* \otimes V$. The evaluation map $V^* \times V \rightarrow \mathbb{C}$ sends $(\lambda, v) \rightarrow \lambda(v)$ with $\lambda \in V^*$ and $v \in V$. This map is bilinear so by the universal property of \otimes induces a linear map $V^* \otimes V \rightarrow \mathbb{C}$.

If we identify $\text{End}(V) = V^* \otimes V$, this is the trace.

The trace in terms of dual bases

Let v_i be a basis of V , and let v_i^* be the dual basis of V^* . Then for $T \in \text{End}(V)$

$$\text{tr}(T) = \sum_i v_i^*(Tv_i).$$

Indeed, let us write $T = (t_{ij})$ as a matrix with the basis v_j . Then

$$\sum_i v_i^*(Tv_i) = \sum_i \sum_j v_i^*(t_{ij}v_j) = \sum_i t_{ii},$$

which is the trace.

Traces of direct sum and tensor product

Let V_1 and V_2 be vector spaces of dimensions d_1 and d_2 . Then we may create vector spaces $V_1 \oplus V_2$ and $V_1 \otimes V_2$ of dimensions $d_1 + d_2$ and $d_1 d_2$. If $T_1 : V_1 \rightarrow V_1$ and $T_2 : V_2 \rightarrow V_2$ are linear transformations, then we have transformations

$$T_1 \oplus T_2 \in \text{End}(V_1 \oplus V_2), \quad T_1 \otimes T_2 \in \text{End}(V_1 \otimes V_2).$$

Their traces are $\text{tr}(T_1) + \text{tr}(T_2)$ and $\text{tr}(T_1) \text{tr}(T_2)$.

Computation for the tensor product

To see that the trace of $T_1 \otimes T_2$ is $\text{tr}(T_1) \text{tr}(T_2)$ let v_i be a basis of V_1 and u_k be a basis of V_2 . Let the matrices of T_1 and T_2 be (a_{ij}) and (b_{kl}) so

$$T_1 v_i = \sum_j a_{ij} v_j, \quad T_2 u_k = \sum_l b_{kl} u_l.$$

With respect to the basis $v_i \otimes u_k$ of $V_1 \otimes V_2$ the matrix of $T_1 \otimes T_2$ is $a_{ij} b_{kl}$, that is

$$(T_1 \otimes T_2) v_i \otimes u_k = \sum_{j,l} a_{ij} b_{kl} (v_j \otimes u_l)$$

so the trace

$$\text{tr}(T_1 \otimes T_2) = \sum_{i,k} a_{ii} b_{kk} = \text{tr}(T_1) \text{tr}(T_2).$$

(A more functorial proof is also possible.)

The character of a representation

Let $\pi : G \rightarrow \text{GL}(V)$ be a representation. The **character** of π is the function

$$\chi_\pi(g) = \text{tr}(\pi(g)).$$

A function $f : G \rightarrow \mathbb{C}$ is a **class function** if $f(gxg^{-1}) = f(x)$, so that it is constant on conjugacy classes. The characters of the irreducible representations of G are important data that carry a lot of information about G .

Since the characters are constant on conjugacy classes, they are data that are related to the conjugacy classes of the group.

Linear characters

Since $GL(1, \mathbb{C}) \cong \mathbb{C}^\times$, a one-dimensional representation $\pi : G \rightarrow GL_1(\mathbb{C})$ is really just a homomorphism $G \rightarrow \mathbb{C}^\times$. Taking the trace map $\text{Mat}_1(\mathbb{C}) \rightarrow \mathbb{C}$ is the trivial operation of extracting the unique coefficient from a 1×1 matrix so in this case there is no real distinction between the representation and its character.

Such a homomorphism $G \rightarrow \mathbb{C}^\times$ is often called a **character** but since characters of higher dimensiona representations are not of this type, we will choose our terminology carefully and call it a **linear character**.

The trivial representation

Every group G has a linear character, namely the character of the **trivial representation** $1_G : G \rightarrow GL(1, \mathbb{C})$ that maps every element of G to the identity.

The character of the trivial representation is the function that has value 1 on every conjugacy class.

Other linear characters can be found by determining the abelian group G/G' where G' is the derived group (commutator subgroup). Any linear character of this abelian group can be pulled back to get a linear character of G .

Example: S_3

The symmetric group S_3 has two linear characters, namely the character of the trivial representation, and also the **sign character** which is the homomorphism $S_3 \rightarrow \{\pm 1\} \subseteq \mathbb{C}^\times$ that is $+1$ on even permutations, and -1 on odd permutations. There is one other two-dimensional irreducible representation π .

To describe the characters we pick representatives of the conjugacy classes, and tabulate the values of the characters.

This is the most important information about this group.

	1	(123)	(12)
trivial rep'n	1	1	1
sign rep'2	1	1	-1
χ_π	2	-1	0

Irreducible characters

If π is a representation (whether irreducible or not) we will just refer to χ_π as a **character**. If π is an irreducible representation we will call χ_π an **irreducible character**. Eventually we will prove that there are only a finite number of irreducible characters. Since every representation can be decomposed as the direct sum of irreducible representations, every character is a sum of irreducible characters.

The character ring

Proposition

Let χ_1 and χ_2 be characters. Then $\chi_1 + \chi_2$ and $\chi_1\chi_2$ are characters.

Indeed, they are the characters of $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ acting on $V_1 \oplus V_2$ and $V_1 \otimes V_2$.

Let us call a class function a **virtual character** or a **generalized character** if it is the difference of two characters. Since the set of characters is closed under addition and multiplication, the virtual characters form a ring, called the **character ring**.

The contragredient or dual representation

If $\pi : G \rightarrow \text{GL}(V)$ is a representation, we also have a representation on the dual space V^* . We recall that the dual space is a contravariant functor, so if $T : V \rightarrow W$ is a linear transformation, the **adjoint** $T^* : W^* \rightarrow V^*$ is defined by $T^*(\lambda) = \lambda \circ T$ for $\lambda \in W^*$. Composition is reversed, so $(T \circ U)^* = U^* \circ T^*$.

Thus we can define $\hat{\pi} : G \rightarrow \text{GL}(V^*)$ by $\hat{\pi}(g) = \pi(g^{-1})^*$. The use of g^{-1} here is needed so that

$$\hat{\pi}(g \circ h) = \hat{\pi}(g) \circ \hat{\pi}(h).$$

The character of the contragredient representation

Proposition

The character of $\hat{\pi}$ is the complex conjugate of the character of π .

To prove this, note that the eigenvalues of $\pi(g)$ are roots of unity, since G has finite order. If $g^n = 1$ then $\pi(g)^n = 1$, so the eigenvalues $\varepsilon_1, \dots, \varepsilon_d$ of $\pi(g)$ satisfy $\varepsilon_i^n = 1$. Now the eigenvalues of $\hat{\pi}(g)$ are the eigenvalues of $\pi(g^{-1})$, since a matrix and its adjoint have the same trace. Thus

$$\chi_{\hat{\pi}}(g) = \sum_i \varepsilon_i^{-1} = \sum_i \bar{\varepsilon}_i = \overline{\chi(g)}.$$

We have used the fact that the inverse of a root of unity is its complex conjugate.

Schur's Lemma part I

Let R be a ring. By an R -module we will mean a left module, unless otherwise stated. Recall that a simple module M is one that has no submodules except M itself and 0 .

Proposition (Schur's Lemma, Part I)

Let M and N be simple R -modules. If $\phi : M \rightarrow N$ is an R -module homomorphism, either $\phi = 0$ or ϕ is an isomorphism.

To prove this, note that $\ker(\phi)$ is a submodule of M , which is not M if $\phi \neq 0$. Since M is simple, $\ker(\phi) = 0$ proving ϕ is injective. Similarly $\text{im}(\phi)$ is a submodule of N , which cannot be 0 since $\phi \neq 0$; since N is simple, $\text{im}(\phi) = N$, proving that ϕ is surjective. Thus ϕ is an isomorphism.

Schur's Lemma, Part II

Proposition (Schur's Lemma, Part II)

Suppose that R is an algebra over an algebraically closed field F and that M is a simple R -module that is finite-dimensional as an F -vector space. If $\phi \in \text{End}_R(M)$ then there exists a scalar $\lambda \in F$ such that $\phi(x) = \lambda x$ for all $x \in M$.

To prove this, since M is a finite-dimensional vector space over an algebraically closed field F , the linear transformation ϕ has an eigenvalue $\lambda \in F$. Now consider the transformation $\phi - \lambda I_M$. This transformation is not injective, so by the first part of Schur's Lemma, it is the zero map.

Variant of Schur's Lemma Part II

There is a variant of this which does not assume that R is an algebra over an algebraically closed field. Remember that a **division ring** is a ring in which every nonzero element has a multiplicative inverse.

Proposition

Suppose that R is a ring and that M is a simple R -module. Then $\text{End}_R(M)$ is a division ring.

Indeed, every module homomorphism that is nonzero is an isomorphism by Schur's Lemma, Part I, so it has an inverse map, which serves as an inverse in the ring $\text{End}_R(M)$.

Hilbert space

Let $L^2(G)$ be the space of functions on G . It is a Hilbert space with inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

The subspace $L^2_{\text{class}}(G)$ of class functions on G inherits this inner product. If π is a representation of G , then $\chi_\pi \in L^2_{\text{class}}(G)$.

Schur orthogonality

On Thursday we will prove the following Theorem:

Theorem

The characters of irreducible representations of G are an orthonormal basis of $L^2_{\text{class}}(G)$. Thus if π_1 and π_2 are irreducible representations with characters χ_1 and χ_2 then

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2, \\ 0 & \text{otherwise.} \end{cases}$$