Lecture 6: Maschke's Theorem

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Group representation theory

Group representation theory is an analog of Fourier analysis. It is a very powerful tool for studying finite groups, or infinite groups such as Lie groups, or even more general things. We will restrict ourselves to finite groups.

A representation of a finite group G is a homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$ where V is a finite-dimensional vector space over a field F. In this class we will always take $F = \mathbb{C}$.

Equivalently, if we choose a basis of V then $\mathrm{GL}(V) \cong \mathrm{GL}_n(F)$ where $n = \dim(V)$, so we could just as easily consider a homorphism $G \longrightarrow \mathrm{GL}_n(F)$. We call such a homomorphism a matrix representation.

Algebras

If F is a field, an algebra is a ring R with an injective ring homomorphism $\phi : F \longrightarrow R$ such that $f(F)$ is contained in the center of R.

For example, if F is a field, and K is a larger field, then K is an algebran over F. The ring $\mathrm{Mat}_n(F)$ is an algebra over F. The quaternions are an algebra over $\mathbb{R}$ but not over $\mathbb{C}$. (Why not?)

The Group Algebra

We may associate with G a ring $\mathbb{C}[G]$ called the group algebra. This is the free vector space on G, so as a vector space it has dimension $|G|$, and a typical element has the form

$$\sum_{g\in G} a_g \cdot g, \qquad a_g \in \mathbb{C}.$$

We have to define the multiplication. The multiplication map $G \times G \longrightarrow G$ extends uniquely to a bilinear map $\mathbb{C}[G] \times \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$. Thus

$$\left(\sum_{g\in G} a_g \cdot g\right)\left(\sum_{g\in G} b_g \cdot g\right) = \sum_{g_1,g_2} a_{g_1} b_{g_2} \cdot g_1 g_2$$

Representations and $\mathbb{C}[G]$ modules

Proposition

The following two data are equivalent:

- *A representation* $\pi : G \longrightarrow \mathrm{GL}(V)$*;*
- *A* $\mathbb{C}[G]$*-module.*

First suppose that $\pi : G \longrightarrow \mathrm{GL}(V)$ is given. We make V into a $\mathbb{C}[G]$-module by

$$\left(\sum_{g \in G} a_g \cdot g \right) v = \sum_{g \in G} a_g \, \pi(g) v.$$

Using the fact that $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, it is easy to check that V becomes a $\mathbb{C}[G]$-module.

A $\mathbb{C}[G]$-module gives a representation

On the other hand, given a $\mathbb{C}[G]$-module V, define $\pi : G \longrightarrow \mathrm{GL}(V)$ by

$$\pi(g)v = g \cdot v.$$

It is easy to see this is a representation. These two constructions are inverses of each other.

Simple modules

Let R be a ring, V an R-module. We say V is simple or irreducible if $V \neq 0$ and V has no proper, nontrivial submodules. For example, if R is an algebra, then every R-module is a vector space (since $F \subseteq R$) and if V is one-dimensional over F, then it is irreducible. However there can be irreducible modules of dimension greater than 1, as we will see.

Projections

Proposition

Let R be a ring, and M an R-module. Suppose that $p : M \longrightarrow M$ is an R-module homomorphism such that $p^2 = p$. Then

$$M = \ker(p) \oplus \operatorname{im}(p)$$

is a decomposition into R-submodules.

Such an endomorphism (satisfying $p^2 = p$) is called a projection. Note that since p is an R-module homomorphism, $\ker(p)$ and $\operatorname{im}(p)$ are both submodules. First let us check that $M = \ker(p) + \operatorname{im}(p)$. Indeed, if $x \in M$ write $x = y + z$ where $y = x - p(x)$ and $z = p(x)$. We have $p(y) = p(x) - p^2(x) = 0$ since $p^2 = p$. Thus $x \in \ker(p)$. Clearly $z \in \operatorname{im}(p)$, showing $M = \ker(p) + \operatorname{im}(p)$.

Projections (continued)

To show the sum is direct, we must show $\ker(p) \cap \operatorname{im}(p) = 0$. Let $x \in \ker(p) \cap \operatorname{im}(p)$. Since $x \in \operatorname{im}(p)$ write $x = p(t)$ for some p. Then since $x \in \ker(p)$, we have

$$0 = p(x) = p(p(t)) = p(t) = x$$

since $p^2 = 0$. Since $M = \ker(p) + \operatorname{im}(p)$ and $\ker(p) \cap \operatorname{im}(p) = 0$, M is the internal direct sum of $\ker(p)$ and $\operatorname{im}(p)$.

Completely reducible modules

A module V is called completely reducible or semsimple if it is a direct sum of simple modules.

Lemma

Suppose that R is an algebra and V a finite-dimensional module such that for every submodule U there exists a submodule W such that $V = U \oplus W$. Then V is completely reducible.

Indeed, if V is 0, this is vacuously true. If not, let U be a nonzero submodule of smallest dimension. Then obviously U must be simple. Find a submodule W such that $V = U \oplus W$. Since $\dim(W) < \dim(V)$, arguing by dimension, W is a direct sum of simple modules. Hence so is V.

Maschke's theorem

Theorem (Maschke)

Let V be a module over $\mathbb{C}[G]$ that is finite-dimensional over $\mathbb{C}$. Then V is completely reducible.

By the Lemma, it is enough to show that a submodule U of V is complemented, that is, there is a submodule W such that $V = U \oplus W$. At least it is obvious that there is a vector subspace W_0 such that $V = U \oplus W_0$; we have to improve this information to find a submodule W such that $V = U \oplus W$.

A projection

Given W_0 we have a map $p_0 : V \longrightarrow V$ whose image is U, defined as follows. Write $x \in V$ uniquely as $x = u + w_0$ with $u \in U$ and $w_0 \in W$. Then define $p_0(x) = u$. The map p_0 is just a linear transformation: it is not an $\mathbb{C}[G]$-module homomorphism, unless we are lucky and W_0 happens to be a submodule.

The of p_0 is contained in U by construction. Moreover if $x \in U$, then $p_0(x) = x$ since in the decomposition $x = u + w_0$ we have $u = x$ and $w_0 = 0$.

Now we claim p_0 is a projection, meaning $p_0^2 = p_0$. Indeed, if $x \in V$ then $p_0(x) \in U$ so $p_0^2(x) = p_0(x)$.

Averaging the projection

The idea is to modify the projection p_0, which is only a linear transformation, to make it into a $\mathbb{C}[G]$-module homomorphism. Define

$$p(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gx).$$

We will show that $p(x)$ is a G-module homomorphism, and that it is a projection with image U.

First let us check that $p(x)$ has image U. Each term $p_0(gx)$ is in U so $g^{-1}p(gx) \in U$ and we concluded that $p(x) \in U$. Therefore $p(V) \subseteq U$.

In which p is the identity on U

Next we show that if $x \in U$ then $p(x) = x$. Indeed, since U is a $\mathbb{C}[G]$-submodule we have $gx \in U$ for all $g \in G$. Therefore $p_0(gx) = gx$ and so $g^{-1}p_0(gx) = x$. Thus

$$p(x) = \frac{1}{|G|}\sum_{g \in G} g^{-1}p_0(gx) = \frac{1}{|G|}\sum_{g \in G} x = x.$$

The averaged projection is a $\mathbb{C}[G]$-module homomorphism

Now p is a projection since if $x \in V$ so $p(x) \in U$ so $p^2(x) = p(x)$.

It is important that $p(x)$ is a $\mathbb{C}[G]$-module homomorphism. To check that $p(rx) = rp(x)$ for $r \in \mathbb{C}[G]$ it is enough to check the case $r = g \in G$. Then

$$p(gx) = \frac{1}{|G|}\sum_{h\in G} h^{-1}p_0(hgx).$$

With g fixed, reindex the sum by replacing h by hg^{-1}. We obtain

$$p(gx) = \frac{1}{|G|}\sum_{h\in G} gh^{-1}p_0(hx) = gp(x).$$

Now p is a projection onto U that is a G-module homomorphism. Therefore $V = U \oplus W$ where $W = \ker(U)$.

Reformulation for representations

We have now proved Maschke's theorem. Let us reformulate it in terms of representations. If $\pi : G \longrightarrow \mathrm{GL}(V)$ is a representation, then an invariant subspace is a vector subspace $U \subseteq V$ such that $\pi(g)U \subseteq U$ for all $g \in G$. Of course, this is the same as a $\mathbb{C}[G]$-submodule.

The representation is irreducible if $V \neq 0$ but the only invariant subspaces are 0 and V itself. Of course, this is the same as a simple $\mathbb{C}[G]$-module.

Theorem (Maschke)

Let $p : G \longrightarrow \mathrm{GL}(V)$ be a representation. If $U \subseteq V$ is an invariant subspace, then there exists another invariant subspace W such that $V = U \oplus W$. Every finite-dimensional representation is a direct sum of irreducible representations.