Lecture 5: More on tensors

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Free modules

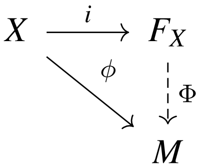
We will consider objects described by universal properties. We will give two examples: free modules over a ring, and tensor product of modules over a commutative ring such as a field. Let R be a ring, and X a set. The free-module F_X may be defined by its universal property which is Theorem 6 in [DF] Section 10.3. This comes equipped with a map $i : X \longrightarrow F_X$ which is part of the data characterizing the free module.

Definition (The Universal Property of the Freem Module)

If M is any R-module and $\phi : X \longrightarrow M$ is any map, then there is a unique R-module homomorphism $\Phi : F_X \longrightarrow M$ such that $\phi = \Phi \circ i$.

The universal property diagrammed

If M is any R-module and $\phi : X \longrightarrow M$ is any map, then there is a unique R-module homomorphism $\Phi : F_X \longrightarrow M$ such that $\phi = \Phi \circ i$.

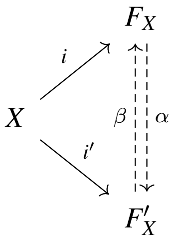


The universal property is a valid definition

Let us note that F_X is characterized up to isomorphism by this property, so we can use it to define the rree module.

If F'_X and $i' : X \longrightarrow F'_X$ is another free-module satisfying the same universal property, then using the universal property for F_X there is an R-module homomorphism $\alpha : F_X \longrightarrow F'_X$ such that $i' = \alpha \circ i$. Using the universal property for F'_X there is a homomorphism $\beta : F'_X \longrightarrow F_X$ such that $i = \beta \circ i'$. We claim that α and β are inverse homomorphisms. To see this, note that $\beta \circ \alpha \circ i = \beta \circ i' = i$. Now $\beta\alpha$ and 1_{F_X} (the identity map) are both homomorphisms $F_X \longrightarrow F_X$ such that $\beta\alpha \circ i = i = 1_{F_X}$. The universal property implies that there is a unique homomorphism $\lambda : F_X \longrightarrow F_X$ such that $\lambda \circ i = i$ and so $\beta\alpha = 1_{F_X}$. Similarly $\alpha\beta = 1_{F'_X}$. So α and β are inverse homomorphisms.

The universal property is a valid definition (continued)



- The universal property of F_X produces α
- The universal property of F'_X produces β
- The uniquess in the universal property shows $\beta\alpha = 1_{F_X}$

Functors

Let us recall the notion of a functor. A functor is like a "homomorphism of categories," though that statement is not strictly correct, only suggestive.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ is a rule that associates to every object A of $\mathcal{C}$ an object $\mathcal{F}A$ of $\mathcal{D}$, and also if $f : A \longrightarrow B$ is a morphism in the category $\mathcal{C}$, there is a morphism $\mathcal{F}f : \mathcal{F}A \longrightarrow \mathcal{F}B$ in the category $\mathcal{D}$.

It is assumed that if $1_A \in \mathrm{Hom}(A, A)$ is the identity morphism then $\mathcal{F}1_A = 1_{\mathcal{F}A}$ and the functor respects compositions in the sense that if $f : A \longrightarrow B$, $g : B \longrightarrow C$ then $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$.

The free module is a functor

Let us show that the free-module is a functor.

For every set X, the free module is only characterized up to isomorphism by the universal property, but we pick a particular realization $i_X : X \longrightarrow F_X$. Thus $\mathcal{F}X = F_X$ is supposed to be a functor. If $f : X \longrightarrow Y$ is a map of sets, we need to define a map $\mathcal{F}f : F_X \longrightarrow F_Y$. We obtain this by use of the universal property. From the map $i_Y \circ f : X \longrightarrow F_Y$ the universal property produces a homorphism $\mathcal{F}f : F_X \longrightarrow F_Y$, which is the unique homomorphism such that $\mathcal{F}f \circ i_X = i_Y \circ f$. It is easy to see that $\mathcal{F}$ is a functor from the category of sets to the category of R-modules.

The free module is a functor (continued)

Showing how given a map $f : X \to Y$ the universal propery of F_X produces an R-module homomorphism $F_X \to F_Y$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ F_X & \overset{\mathcal{F}f}{\dashrightarrow} & F_Y \end{array}$$

Denoting this homomorphism $\mathcal{F}f : F_X \to F_Y$, the free module becomes a functor $\mathcal{F}$ from the category of sets to the category of R-modules.

Universal properties as initial objects in a category

In Lecture 1 we defined the notion of an initial object in a category, and showed that any two initial objects are isomorphic. This argument is formally very similar to the above. (Please review.)

Problem

Can you deduce the uniqueness of the free module (up to isomorphism) from the uniqueness of the initial object? The problem is to define the right category.

Contravariant functors

What we have defined is sometimes called a covariant functor. There are also contravariant functors. For a contravariant functor, the directions of arrows is reversed. Thus if $\mathcal{G}$ is a contravariant functor and $f : A \longrightarrow B$ is a morphism in the category $\mathcal{C}$ then $\mathcal{F}f : \mathcal{F}B \longrightarrow \mathcal{F}A$ and of course $\mathcal{F}(g \circ f) = \mathcal{F}f \circ \mathcal{F}g$ when the composition $g \circ f$ is defined.

An example of a contragredient functor is the dual vector space. This is a functor from the category $\mathcal{V}$ of finite-dimensional vector spaces over a field F to itself. To see that this functor is contragredient suppose $f : V \longrightarrow W$ is a homomorphism. Then define $f^* : W^* \longrightarrow V^*$ to be composition with f. Thus if $\lambda \in W^*$ so λ is a functional $W \longrightarrow F$ then $f^*(\lambda) = \lambda \circ f \in V^*$.

Hom sets as functors

Hom sets are functors in both variables. Let us consider first now $\operatorname{Hom}(A,B)$ is functorial in B.

For definiteness, let us consider the category of R-modules. With an object A fixed, $\operatorname{Hom}(A,B)$ is functorial in B. That is, if $f : B \longrightarrow B'$ is any homomorphism, then composition with f is a functor from the category of R-modules to the category of sets. Thus $\operatorname{Hom}(A,f)$ is the map $\operatorname{Hom}(A,B) \longrightarrow \operatorname{Hom}(A,B')$ that is composition with f:

$$\operatorname{Hom}(A,f)(g) = gf \in \operatorname{Hom}(A,B'), \qquad g \in \operatorname{Hom}(A,B).$$

For brevity we denote this map $\operatorname{Hom}(A,f) = f_*$.

We denote this functor $\operatorname{Hom}(A,-)$.

Hom is a bifunctor

Similarly if B is a fixed module then $\mathrm{Hom}(-, B)$ is a functor from R-modules to sets, but this functor is contragredient! Please check this. To summarize, $\mathrm{Hom}(A, B)$ is a functor in both A and B. It is covariant in B but contravariant in A.

There is a compatibility between the two functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$. Suppose that $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ are two homomorphisms. Then

$$\begin{array}{ccc} \mathrm{Hom}(A', B) & \xrightarrow{f^*} & \mathrm{Hom}(A, B) \\ \downarrow g_* & & \downarrow g_* \\ \mathrm{Hom}(A', B') & \xrightarrow{f^*} & \mathrm{Hom}(A, B') \end{array}$$

commutes. We call Hom a bifunctor.

References

For tensor product, we will not follow Dummit and Foote, since we will consider only tensor products over commutative rings, mainly a field.

In addition to Brian Conrad's lecture of Thursday, April 16, we can recommend the treatment in Lang's Algebra, which restricts to a commutative ring.

In Lang's Algebra, which usually contains more information about any topic than Dummit and Foote, only tensor products over commutative rings are considered. In my opinion, Dummit and Foote work in more generality than we will need and thereby make the theory more complicated.

Bilinear maps

Assume that the ground ring R is commutative; often we will take $R = F$ to be a field. Let A, B and C be R-modules. A map $\phi : A \times B \longrightarrow C$ is bilinear if

$$\phi(r_1a_1 + r_2a_2, b) = r_1\phi(a_1, b) + r_2(a_2, b),$$

$$\phi(a, r_1b_1 + r_2b_2) = r_1\phi(a, b_1) + r_2\phi(a, b_2).$$

In other words, it is linear in the first variable (if b is fixed) and also linear in the second variable (if a is fixed).

The tensor product over a commutative ring

The tensor product $A \otimes B$ is characterized by a universal property.

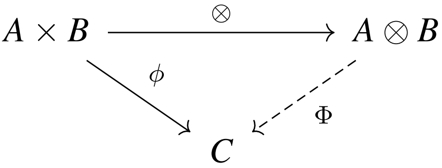
Definition (The universal property of $\otimes$)

- The tensor product is an R-module $A \otimes B$, with a bilinear map $\otimes : A \times B \longrightarrow A \otimes B$.
- Second, if $\phi : A \times B \longrightarrow C$ is any bilinear map, then there is a unique linear map (homomorphism) $\Phi : A \otimes B \longrightarrow C$ such that $\phi = \Phi \circ \otimes$.

We usually write $a \otimes b$ instead of $\otimes(a, b)$.

The universal property diagrammed

As will the free module, the universal propery characterizes the tensor product up to isomorphism.



The proof is the same as for the free module. So the issue is whether a module $A \otimes B$ and a bilinear map $\otimes$ satisfying this property exists.

Existence of the tensor product

In his Thursday. April 16 lecture, Brian proved the existence of the tensor product if $R = F$ is a field, so that modules are vector spaces, which he assumed finite-dimensional.

If R is an arbitrary commutative ring, then a tensor product exists with exactly the same definition. We will not give the proof, but see Lang's Algebra for a proof. The universal property characterizes the tensor product up to isomorphism. Since the plan is to base proofs on the universal property instead of a particular construction, the proof is not so important.

General remarks

Like Hom, The tensor product is a bifunctor. Unlike Hom, it is covariant in both variables.

Our goal is to use the universal property to establish properties. Thus a particular realization of $A \otimes B$ is not as important, and we will not worry about the proof.

For vector spaces, it is clear from the construction in Brian Conrad's lecture that if V, W are vector spaces over F then

$$\dim(V \otimes W) = \dim(V) \dim(W).$$

More precisely if v_i are a basis of V and w_j are a basis of W, then $v_i \otimes w_j$ are a basis of $V \otimes W$, so the dimensions are multiplicative.

Nuances when R is not a field

If F is a field, $\dim(V \otimes W) = \dim(V)\dim(W)$ implies that if V, W are nonzero, so is $V \otimes W$.

On the other hand suppose that $R = \mathbb{Z}$. We will show that there can be two nonzero R-modules whose tensor product is zero!

The Lemma

With $R = \mathbb{Z}$, an R-module is just an abelian group M. Now $\mathbb{Q}$ is an example of an abelian group.

Lemma

Let M be a finite abelian group, regarded as a $\mathbb{Z}$-module. Then for any $\mathbb{Z}$-module A, any bilinear map $\phi : \mathbb{Q} \times M \longrightarrow A$ is the zero map.

Indeed, let $q \in \mathbb{Q}$ and $x \in M$. Since M is finite, $nx = 0$ for some n. Then

$$\phi(q, x) = \phi(nq/n, x) = n\phi(q/n, x) = \phi(q/n, nx) = \phi(q/n, 0) = 0.$$

The example

Proposition

Let M be a finite abelian group, regarded as a $\mathbb{Z}$-module. Then $\mathbb{Q} \otimes_{\mathbb{Z}} M = 0$.

We deduce this from the Lemma we just proved.

Lemma

Let M be a finite abelian group, regarded as a $\mathbb{Z}$-module. Then for any $\mathbb{Z}$-module A, any bilinear map $\phi : \mathbb{Q} \times M \longrightarrow A$ is the zero map.

By the Lemma, the zero module satisfies the universal property of the tensor product, so $\mathbb{Q} \otimes M = 0$. Contrast this with the homework problem that if $R = F$ is a field, then $V \otimes W = 0$ implies $V = 0$ or $W = 0$

The relation between Hom and $\otimes$

We now specialize to the case where the ground ring $R = F$ is a field. Thursday Brian Conrad proved

$$\boxed{\mathrm{Hom}(V, W) \cong V^* \otimes W.}$$

To repeat the proof, V^* consists of linear functionals $\lambda : V \longrightarrow F$. We define a bilinear map $\theta : V^* \times W \longrightarrow \mathrm{Hom}(V, W)$ as follows. If $\lambda \in V^*$, $v \in V$ let $\theta(\lambda, v)$ be the rank one linear map $V \longrightarrow W$ defined by

$$\theta(\lambda, v)(x) = \lambda(x)v.$$

The universal property of $V^* \otimes W$ gives us a homomorphism $\Theta : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$ such that $\Theta(\lambda \otimes v) = \theta(\lambda, v)$. The homomorphism Θ is surjective since any linear transformation in $\mathrm{Hom}(V, W)$ is a sum of rank one transformations. The dimensions are equal, so Θ is an isomorphism.

Natural transformations

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two categories and both $\mathcal{F}$ and $\mathcal{G}$ are functors from $\mathcal{C}$ to $\mathcal{D}$. Suppose that for every object A of $\mathcal{C}$ there is a morphism $\mu_A : \mathcal{F}A \longrightarrow \mathcal{G}A$. The morphism μ_A is called natural if for every morphism $f : A \longrightarrow B$ in the category $\mathcal{C}$ we have $\mu_B \circ (\mathcal{F}f) = (\mathcal{G}f) \circ \mu_A$, so the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}A & \xrightarrow{\mathcal{F}f} & \mathcal{F}B \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} \\ \mathcal{G}A & \xrightarrow{\mathcal{F}f} & \mathcal{G}B \end{array}$$

There is a similar notion of naturality if $\mathcal{F}$ and $\mathcal{G}$ are contravariant. But they must both be either covariant or contravariant.

The $\mathrm{Hom} - \otimes$ relation is natural

We recall that the bifunctor $\mathrm{Hom}(V, W)$ is contragredient in V and covariant in W, whereas $V \otimes W$ is covariant in both variables. To line them up we apply the contragredient dual space functor to V. This leads to the isomorphism

$$\Theta : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$$

which we have already proved. This isomorphism is natural in both V and W.

Natuality in W

Let us fix V and let $g : W \longrightarrow W'$ be a linear transformation. Then naturality means that we have a commutative diagram:

$$\begin{array}{ccc} V^* \otimes W & \xrightarrow{\Theta} & \mathrm{Hom}(V, W) \\ \Big\downarrow{\scriptstyle 1_{V^*} \otimes g} & & \Big\downarrow{\scriptstyle g_*} \\ V^* \otimes W' & \xrightarrow{\Theta} & \mathrm{Hom}(V, W') \end{array}$$

By the universal property, it is enough to check

$$\begin{array}{ccc} V^* \times W & \xrightarrow{\theta} & \mathrm{Hom}(V, W) \\ \Big\downarrow{\scriptstyle 1_{V^*} \times g} & & \Big\downarrow{\scriptstyle g_*} \\ V^* \otimes W' & \xrightarrow{\theta} & \mathrm{Hom}(V, W') \end{array}$$

Both compositions send (λ, w) to $v \mapsto \lambda(v)g(w)$.

Naturality in V

The isomorphism Θ is also natural in V^*. We have already observed that both functors $V^* \otimes W$ and $\mathrm{Hom}(V, W)$ are contragredient in V^*. So if $f : V \longrightarrow V'$ is a homomorphism, the commutativity we need has the form

$$\begin{array}{ccc} (V')^* \otimes W & \xrightarrow{\Theta} & \mathrm{Hom}(V, W) \\ \downarrow{\scriptstyle f^* \otimes 1_W} & & \downarrow{\scriptstyle f^*} \\ V^* \otimes W & \xrightarrow{\Theta} & \mathrm{Hom}(V, W) \end{array}$$

This is possible because both $V^* \otimes W$ and $\mathrm{Hom}(V, W)$ are contragredient in V.