$N_1 \cap N_2$

$$\begin{tikzcd}
 & M \arrow[d, dash] & \\
 & N_1+N_2 \arrow[dl, dash] \arrow[dr, dash] & \\
N_1 \arrow[dr, dash] & & N_2 \arrow[dl, dash] \\
 & N_1 \cap N_2 &
\end{tikzcd}$$

LATTICE DIAGRAM

$$\begin{tikzcd}
A \arrow[d, dash] \\
B
\end{tikzcd}$$

MEANS $B \subseteq A$

$$N_1 + N_2 = \{ n_1 + n_2 \mid n_i \in N \}$$

SMALLEST SUBMODULE CONTAINING BOTH.

SUPPOSE $S \subseteq M$ IS ANY SET

$\langle S \rangle$ IS THE "SUBMODULE GENERATED BY S"

I.E. THE SMALLEST SUBMODULE THAT CONTAINS S.

$$\langle S \rangle = \bigcap_{\substack{\text{SUBMODULES} \\ U \text{ OF } M \\ \text{S.T. } S \subset U}} U .$$

$$\langle N_1 \cup N_2 \rangle = N_1 + N_2 .$$

SOMETIMES WE CONSIDER THE SUM $N_1 + N_2$ DIRECT.

IF $N_1 \cap N_2 = 0$ WE WRITE

$N_1 \oplus N_2$.

EXPLANATION. IF N_1 AND N_2 ARE ABSTRACT R-MODULES WE CAN CONSTRUCT THE CARTESIAN PRODUCT $N_1 \times N_2$ AND THIS IS AN R-MODULE. IF N_1, N_2 ARE SUBMODULES OF M AND $N_1 \cap N_2 = 0$ THEN $N_1 + N_2 \cong N_1 \times N_2$.

BECAUSE $\phi : N_1 \times N_2 \to N_1 + N_2$

$\phi(n_1, n_2) = n_1 + n_2$ IS AN ISOMORPHISM. IT IS INJECTIVE SINCE $\ker(\phi) = \{(n_1, n_2) \mid n_1 + n_2 = 0\}$

IF $(n_1, n_2) \in \ker$ $\quad n_1 = -n_2 \in N_1 \cap N_2 = 0$

ϕ IS INJECTIVE AND SURJ. BY CONSTRUCTION. NOTATION:

$N_1 \oplus N_2$ IS USED IN THIS CIRCUMSTANCE.

$N_1 \oplus N_2$ WOULD ALSO BE USED FOR $N_1 \times N_2$ IF N_1, N_2 ARE ABSTRACT R-MODULES.

FOR FINITE NUMBERS OF R-MODULES

DIRECT SUM AND DIRECT PRODUCT ARE ISOMORPHIC.

$\uparrow$

ABSTRACT

CONSIDER A POSSIBLY INFINITE FAMILY OF ABSTRACT MODULES WE CAN CONSTRUCT TWO MODULES

$$\prod_{i \in I} N_i . \qquad \bigoplus_{i \in I} N_i .$$

(DIRECT PRODUCT) (DIRECT SUM)

$N_i \ (i \in I)$

THESE ARE NOT THE SAME IF $|I| = \infty$.

$$\prod_{i \in N_i} N_i = \{ (n_i) \mid n_i \in N_i \}$$

$$I \to \bigcup N_i$$
$$i \mapsto n_i \in N_i$$

OBVIOUS + AND R-MODULE STRUCTURE

$\bigoplus N_i \subset \prod N_i$

$\parallel$

$\{(n_i) \mid n_i = 0 \text{ FOR ALMOST ALL } i\}.$

$(n_i) + (n_i') = (m_i)$

$m_i = n_i + n_i'.$

$n_i, n_i' \in N_i.$

HOMOMORPHISMS

$p_j : \prod N_i \longrightarrow N_j$

$p_i((n_i \mid i \in I)) = n_j.$

"UNIVERSAL PROPERTY".

GIVEN ANY MODULE M AND MAPS $M \xrightarrow{f_i} N_i$

$$f_i : M \longrightarrow N_i$$

THERE IS A UNIQUE HOMOMORPHISM

$$F : M \rightsquigarrow \prod N_i$$

SUCH THAT

$$\begin{tikzcd}
M \arrow[rr, dashed, "F"] \arrow[dr, "f_i"'] & & \prod N_i \arrow[dl, "p_i"] \\
& N_i &
\end{tikzcd}$$

WHAT IS F: $F(m) = (f_i(m) \mid i \in I)$

$$p_i(F(m)) = f_i(m)$$

IMPORTANT THAT F IS UNIQUE.

WE CAN ALSO MAKE A UNIVERSAL PROPERTY OF $\bigoplus N_i$.

THERE ARE $\iota_j : N_j \longrightarrow \bigoplus N_i$

$$\iota_j(n_j) = (0, 0, \dots, n_j, 0, \dots)$$
$$= (n_i) \quad n_i = 0 \text{ IF } i \neq j.$$

$$\bigoplus N_i = \{(n_i)_{i \in I}\}$$

$$\text{SUPP}(n_i) = \{i \in I \mid n_i \neq 0\}$$

IS FINITE.

UNIVERSAL PROPERTY:

IF B IS AN R-MODULE WITH HOMOMORPHISMS $\beta_j : N_j \to B$ THERE EXISTS A UNIQUE HOMOMORPHISM

$$\bigoplus N_i \xrightarrow[\beta]{} B$$

SUCH THAT $\beta_j = \beta \circ \iota_j$

$$\begin{tikzcd}
N_j \arrow[rr, "\iota_j"] \arrow[dr, "\beta_j"'] & & \bigoplus N_i \arrow[dl, dashed, "\beta"] \\
& B &
\end{tikzcd}$$

$$\beta\left((n_i)\right) = \sum_{\substack{i \in I \\ n_i \neq 0}} \beta_i(n_i)$$

FINITE SUM.

IF $I = \{1, 2\}$ THEN

$$N_1 \times N_2 \cong N_1 \oplus N_2$$

$$\begin{tikzcd}
N_1 \arrow[dr, shift right, "i_1"'] & & N_2 \arrow[dl, shift left, "i_2"] \\
& N_1 \oplus N_2 \arrow[ul, shift right, "p_1"'] \arrow[ur, shift left, "p_2"] &
\end{tikzcd}$$

$i_1(n_1) = (n_1, 0)$ $\quad$ $i_2(n_2) = (0, n_2)$

$p_1(n_1, n_2) = n_1$, $\quad$ $p_2(n_1, n_2) = n_2$

$$i_1 p_1 + i_2 p_2 = 1_{N_1 \oplus N_2}.$$

WE'LL USE THIS SOON TO PROVE

$$M \otimes (N_1 \oplus N_2) = (M \otimes N_1) \oplus (M \otimes N_2)$$

FREE MODULES.

SUPPOSE M IS AN R-MODULE AND WE HAVE A SUBSET $m_i \in M$ $(i \in I)$. SUPPOSE EVERY ELEMENT OF M CAN BE WRITTEN UNIQUELY AS

$$\sum_{i \in I} r_i m_i \,, \quad r_i \in R$$

$r_i = 0$ FOR ALMOST ALL i.

THEN m_i IS A **BASIS** OF M AND M IS A *FREE MODULE*

AN ELT $m \in M$ IS CALLED TORSION IF $rm = 0$ SOME $r \neq 0$.

THE TORSION ELTS $\mathrm{TOR}(M)$ FORM A SUBMODULE. IF M IS FREE $\mathrm{TOR}(M) = 0$.

FOR EXAMPLE $\mathbb{Z}/2\mathbb{Z}$ IS NOT FREE SINCE IT HAS TORSION.

$\mathrm{TOR}(M)$ IS A MODULE IF R IS AN INTEGRAL DOMAIN. (NO ZERO DIVISORS.)

IF $x, y \in \mathrm{TOR}(M)$

$rx = 0$

$r \neq 0$

$sy = 0$

$a \neq 0$

$\Rightarrow ra(x+y) = 0$

$ra \neq 0$.

THE FREE MODULE M

$$M \cong \bigoplus_{i \in I} R$$

$$\sum r_i m \mapsto (r_i) \qquad r_i \in R$$

UNIVERSAL PROPERTY OF FREE MODULE.

SUPPOSE I IS A SET AND WE HAVE A MODULE F WITH

A MAP $\alpha : I \longrightarrow F$. THEN

ASSUME THAT WHENEVER

T IS AN R-MODULE AND

$f : I \longrightarrow T$ IS ANY MAP

THERE EXISTS A UNIQUE ! HOMOMORPHISM

$\phi : F \longrightarrow T$ SUCH THAT $f = \phi\alpha$.

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd}$$

$\alpha(i) = m_i$ BASIS VECTOR.

BASIS OF F

$$\phi\left(\sum r_i m_i\right) = \sum r_i f(m_i)$$

TYPICAL ELT OF F

$r_i = 0$ ALMOST ALL i

NEXT WEEK: UNIVERSAL PROPERTIES ARE CATEGORICAL.

READ D&F 19.1 – 19.3.

O.H. TOMORROW 12:30 - 2:30.