# Lecture 14. Frobenius Groups (II)

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## Review: Frobenius Groups

### Definition

A Frobenius Group is a group $G$ with a faithful transitive action on a set $X$ such that no element fixes more than one point.

An action of $G$ on a set $X$ gives a homomorphism from $G$ to the group of bijections $X$ (the symmetric group $S_{|X|}$). In this definition faithful means this homomorphism is injective. Let $H$ be the stabilizer of a point $x_0 \in X$. The group $H$ is called the Frobenius complement.

Today we will prove:

### Theorem (Frobenius (1901))

*A Frobenius group $G$ is a semidirect product. That is, there exists a normal subgroup $K$ such that $G = HK$ and $H \cap K = \{1\}$.*

## Review: The mystery of Frobenius' Theorem

Since Frobenius' theorem doesn't require group representation theory in its formulation, it is remarkable that no proof has ever been found that doesn't use representation theory!

Web links:

- Frobenius groups (Wikipedia)
- Fourier Analytic Proof of Frobenius' Theorem (Terence Tao)
- Math Overflow page on Frobenius' theorem
- Frobenius Groups (I) (Lecture 14)

## The precise statement

Last week we introduced the notion of a Frobenius group. This is a group $G$ that acts transitively on a set $X$ in which no element except the identity fixes more than one point. Let $H$ be the isotropy subgroup of an element, and let be $K^* \cup \{1\}$ where $K^*$ is the set of elements with no fixed points. The group $H$ is called the Frobenius complement and the set $K$ (which will turn out to be a group) is called the Frobenius kernel.

Our goal is to prove Frobenius' theorem.

**Theorem (Frobenius)**

*Let $G$ be a Frobenius group, and let $K$ be the set of elements that either are the identity, or which have no fixed points. Then $K$ is a normal subgroup of $G$.*

## The strategy

On Thursday, we discussed a strategy for proving this. We noted that if the theorem is true, then $G$ is a semidirect product, $G = H \ltimes K$, and so $H \cong G/K$. This implies that any representation of $H$ can be extended to a representation of $G$. So the strategy is to prove this fact directly, without assuming Frobenius' theorem. If we know that any representation of $H$ can be extended to $G$, we can start with a faithful representation $\pi : H \longrightarrow \mathrm{GL}(V)$, extend it to $G$, and show that $K$ is the kernel. This will prove that $K$ is a normal subgroup.

# Generalized characters

It is convenient to work with generalized characters. Recall that a generalized character of a group $G$ is a class function that is the difference between two characters. Since characters are closed under addition and multiplciation, generalized characters form a ring, $X(G)$. It is the ring of class functions that are linear combinations with $\mathbb{Z}$ coefficients of the irreducible characters.

We have defined induction on characters, but it extends by linearity to generalized characters. The induction formula:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(xgx^{-1})$$

and the Frobenius reciprocity law

$$\langle \chi^G, \theta \rangle_G = \langle \chi, \theta \rangle_H, \qquad \chi \in X(H), \theta \in X(G)$$

are true for generalized characters, as is obvious by linearity.

## The Trivial Intersection Property

**Lemma**

*Let $G$ be a Frobenius group and let $H$ be the stabilizer of some $x \in X$. Then if $g \notin H$, we have*

$$H \cap gHg^{-1} = \{1\}.$$

Indeed, if $g \notin H$ then $gx \neq x$. Let $y = gx$. Then $H$ and $gHg^{-1}$ are the stabilizers of the distinct points $x$ and $y$, so by definition of a Frobenius group, the only element of $H \cap gHg^{-1}$ is the identity.

We will refer to this as the trivial intersection property of $H$.

## Extending generalized characters from $H$ to $G$

We will prove:

**Proposition**

*Let $G$ be a Frobenius group acting on the set $X$. Then there is a linear map $\lambda : X(H) \longrightarrow X(G)$ such that if $\chi \in X(H)$ and $\overline{\chi} = \lambda(\chi)$ then $\overline{\chi}(h) = \chi(h)$ for $h \in H$. If $k \in K$ then $\overline{\chi}(k) = \chi(1)$. The map $\lambda$ is an isometry in the sense that*

$$\langle \lambda(\chi), \lambda(\psi) \rangle_G = \langle \chi, \psi \rangle_H.$$

## Definition of $\overline{q}$

Let $X^0(H)$ be the ideal of $X(H)$ consisting of generalized functions that vanish at $1$. Then

$$X(H) = \mathbb{Z} \cdot 1_H \oplus X^0(H),$$

where $1_H$ denotes the character of the trivial representation, which is the constant function equal to $1_H$. Note that $1_H$ is the unit element in the ring $X(H)$.

We will define the map $\lambda$ differently for $1_H$ and for $X^0(H)$.

- If $\chi \in X^0(H)$ we define $\lambda(\chi) = \chi^G$.
- If $\chi = 1_H$ we define $\lambda(1_H) = 1_G$.

## Induction in $X^0(H)$

We need to check that if $\overline{\chi} = \lambda(\chi)$ then $\overline{\chi}$ agrees with $\chi$ on $H$. This is obvious if $\chi = 1_H$ so we may assume $\chi \in X^0(H)$, meaning $\chi(1) = 0$. Then $\overline{\chi} = \chi^G$.

**Lemma**

*If $h \in H$ and $\chi \in X^0(H)$ then*

$$\chi^G(h) = \chi(h).$$

We have

$$\chi^G(h) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(xhx^{-1}).$$

If $h = 1$, this is $0$ since $\chi(1) = 0$. Thus assume $h \neq 1$.

## Proof

Now remember the trivial intersection property of the Frobenius complement:

$$xHx^{-1} \cap H = \begin{cases} H & \text{if } x \in H, \\ \{1\} & \text{if } x \notin H. \end{cases}$$

So

$$\dot{\chi}(xhx^{-1}) = \begin{cases} \chi(xhx^{-1}) & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

In the first case, since $\chi$ is a class function on $H$, $\chi(xhx^{-1}) = \chi(h)$. Therefore we get

$$\chi^G(h) = \frac{1}{|H|} \sum_{x \in H} \chi(h) = \chi(h)$$

when $\chi \in X^0(G)$. This proves the Lemma.

## $\overline{\chi}$ extends $\chi$

The Lemma shows that of $\overline{\chi} = \lambda(\chi)$ then $\overline{\chi}(h) = \chi(h)$ for $h \in H$ provided $\chi \in X^0(H)$. We must also check this if $\chi = 1_H$ and $\lambda(\chi) = 1_G$, but in that case it is clearly true.

We see that if $\chi$ is a generalized character of $H$ then $\lambda(\chi) = \overline{\chi}$ extends $\chi$, that is, agrees with $\chi$ on $H$.

## If $k \in K$ then $\overline{\chi}(k) = \chi(1)$

We also need to know that if $k \in K$ then $\overline{\chi}(k) = \chi(1)$. We handle the cases $\chi = 1_H$ and $\chi \in X^0(H)$ separately. If $\chi = 1_H$ then $\overline{\chi}(k) = 1 = \chi(1)$. If on the other hand $\chi \in X^0(H)$, then

$$\overline{\chi}(k) = \chi^G(k) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(xkx^{-1}).$$

This is zero, since $K$ has no fixed points, and so there is no way of conjugating it into the isotropy subgroup $H$. Thus $\overline{\chi}(k) = 0 = \chi(1)$ for $\chi \in X^0(H)$. We have proved that $\overline{\chi}(k) = \chi(1)$ for all $\chi$.

## The map $\chi \to \overline{\chi}$ is an isometry

To finish the proof of the Proposition, we must show that $\lambda$ is an isometry. Thus, let $\chi, \psi$ be generalized characters of $H$ and $\overline{\chi} = \lambda(\chi)$, $\overline{\psi} = \lambda(\psi)$ their extensions to $G$. We may assume that either $\chi \in X^0(H)$ or $\chi = 1_H$. In the first case, we have $\overline{\chi} = \chi^G$ and by Frobenius reciprocity

$$\left\langle \overline{\chi}, \overline{\psi} \right\rangle_G = \left\langle \chi^G, \overline{\psi} \right\rangle_G = \left\langle \chi, \overline{\psi} \right\rangle_H .$$

For the last inner product, we are restricting $\overline{\psi}$ to $H$, and we have proved that restriction is just $\psi$, so in this case, we've proved

$$\left\langle \overline{\chi}, \overline{\psi} \right\rangle_G = \langle \chi, \psi \rangle_H .$$

## Proof of the isometry, continued

Next we consider the case $\chi = 1_H$, so $\overline{\chi} = 1_G$ by definition.

We can again handle the cases $\psi \in X^0(H)$ and $\psi = 1_H$ separately. The case $\psi \in X^0(H)$ is similar to the case $\chi \in X^0(H)$ and can be handled the same way, using Frobenius reciprocity.

We are left with the case where both $\chi = 1_H$ and $\psi = 1_H$. Then $\overline{\chi} = 1_G$ and $\overline{\psi} = 1_G$. But then

$$\left\langle \overline{\chi}, \overline{\psi} \right\rangle_G = \langle 1_G, 1_G \rangle = 1 = \langle 1_H, 1_H \rangle_H = \langle \chi, \psi \rangle_H.$$

## The lift takes characters to characters

We have proved the Proposition. Let us repeat it:

**Proposition**

*Let $G$ be a Frobenius group acting on the set $X$. Then there is a linear map $\lambda : X(H) \longrightarrow X(G)$ such that if $\chi \in X(H)$ and $\overline{\chi} = \lambda(\chi)$ then $\overline{\chi}(h) = \chi(h)$ for $h \in H$. If $k \in K$ then $\overline{\chi}(k) = \chi(1)$. The map $\lambda$ is an isometry in the sense that*

$$\langle \lambda(\chi), \lambda(\psi) \rangle_G = \langle \chi, \psi \rangle_H.$$

We can extract more information.

**Proposition**

*If $\chi$ is a character of $H$, then $\overline{\chi}$ is a character of $G$. If $\chi$ is an irreducible character of $H$, then $\overline{\chi}$ is an irreducible character of $G$.*

## Proof

The first statement follows from the second, so we may assume that $\chi$ is irreducible. Indeed, we have $\langle \overline{\chi}, \overline{\chi} \rangle_G = \langle \chi, \chi \rangle_H = 1$. Now $\overline{\chi}$ is a generalized character, so we may write $\overline{\chi} = \sum n_i \chi_i$ where $n_i$ are integers. By Schur orthogonality,

$$1 = \langle \overline{\chi}, \overline{\chi} \rangle = \sum n_i^2.$$

The only way $1$ can be written as a sum of squares is if only one of the $n_i$ is nonzero, and that $n_i = \pm 1$. Thus $\overline{\chi}$ is $\pm \chi_i$. We can rule out the possibility that $\overline{\chi} = -\chi_i$ because $\overline{\chi}(1) = \chi(1) > 0$ and the degree $\chi_i(1)$ is also $> 0$.

## The character characterizes the kernel

We have almost everything we need to prove Frobenius' theorem.

### Proposition

*Let* $\pi : G \longrightarrow \mathrm{GL}(V)$ *be a representation, not necessarily irreducible. Then*

$$\ker(\pi) = \{g \in G | \chi(g) = \chi(1)\}.$$

It is obvious that if $g \in \ker(\pi)$, then $\pi(g) = I_V = \pi(1)$ so $\chi(g) = \dim(V) = \chi(1)$. Conversely, suppose that $\chi(g) = \dim(V)$. Let $d = \dim(V)$ and let $\varepsilon_1, \cdots, \varepsilon_d$ be the eigenvalues of $\pi(g)$. Then $|\varepsilon_i| = 1$ but $\varepsilon_1 + \ldots + \varepsilon_d = d$, and by the "converse to the triangle identity" this implies that $\varepsilon_i = 1$. Thus $\pi(g)$ is the identity matrix, so $g \in \ker(\pi)$.

## Proof of Frobenius' theorem

To prove Frobenius' theorem, let $\chi$ be the character of any faithful representation $\pi$ of $H$, for example the regular representation. Then $\overline{\chi}$ is the character of a representation whose kernel contains $K$, since we proved $\overline{\chi}(k) = \chi(1) = \overline{\chi}(1)$ for $k \in K$. The kernel cannot contain any other non-identity element $g$, since $g$ would be in an isotropy subgroup, hence conjugate to non-identity element of $H$; but the kernel does not contain any non-identity element of $H$ since $\pi$ is faithful.

Since the set $K$ has been identified as the kernel of a representation, it is a normal subgroup!

## A look ahead

We have more to say about Frobenius groups, and will resume this topic on Thursday. For the time being let us make an observation, which we will illustrate with the Frobenius group

$$G = \langle h, k | h^4 = k^5 = 1, hkh^{-1} = k^2 \rangle$$

of degree 20 that we computed last week. Recall that this is the group of affine transformations of $\mathbb{F}_5$.

We have employed two methods of using a subgroup to construct characters of a larger group $G$.

- Extend a representation from the subgroup to $G$;
- Induce a representation from the subgroup to $G$.

We've used both these methods in our Frobenius group.

## Extending versus inducing

| | | | | | |
|---|---|---|---|---|---|
| | 1 | 4 | 5 | 5 | 5 |
| | 1 | $k$ | $h$ | $h^2$ | $h^3$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $i$ | $-1$ | $-i$ |
| $\chi_3$ | 1 | 1 | $-1$ | 1 | $-1$ |
| $\chi_4$ | 1 | 1 | $-i$ | $-1$ | $i$ |
| $\chi_5$ | 4 | $-1$ | 0 | 0 | 0 |

- The characters $\chi_1, \cdots, \chi_4$ are extended from irreducible characters of the group $H$.
- The character $\chi_5$ is induced from an irreducible character of $K$.

## The case of a Frobenius group

Let $G$ be a Frobenius group with kernel $K$ and complement $H$.

- Today we proved that every irreducible character of $H$ may be extended to an irreducible character of $G$.
- On Thursday we will prove that if $\tau$ is an irreducible character of $K$, and $\tau \neq 1$, then the induced character $\tau^G$ is irreducible.

Every irreducible of $G$ comes from either a representation of $H$ or a representation of $K$. The playoff between extending and inducing characters of subgroups is a common feature.

What is remarkable about Frobenius groups is that every irreducible of $G$ is either extended from a character of $G$ or induced from an irreducible of $H$.

## Heisenberg groups of order $q^3$

A group $G$ in which the derived group $G'$ is contained in the center $Z(G)$ is called a 2-step nilpotent group or Heisenberg group. The term Heisenberg group comes from an analogy with the Heisenberg commutation relations in quantum mechanics.

For example, let $F = \mathbb{F}_q$ be a finite field, and consider the group of order $q^3$:

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mid x, y, z \in F \right\}.$$

The center equals the derived group:

$$Z(G) = G' = \left\{ \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} \mid x, y, z \in F \right\}.$$

## Central characters

If $G$ is any group and $\pi : G \to \mathrm{GL}(V)$ an irreducible representation, and if $z \in Z(G)$, then $\pi(z)\pi(g) = \pi(g)\pi(z)$ for all $g \in G$. In other words, $\pi(z)$ is a $\mathbb{Z}[G]$-module homomorphism $V \to V$. By Schur's Lemma, this implies that $\pi(z)$ acts by a scalar.

Thus there is a function $\xi : Z(G) \to \mathcal{C}^\times$ such that

$$\pi(z) = \xi(z)\, I_V, \qquad z \in Z(G).$$

Since $\pi(z_1)\pi(z_2) = \pi(z_1 z_2)$ we have

$$\xi(z_1 z_2) = \xi(z_1)\xi(z_2),$$

so $\xi$ is a linear character of $Z(G)$. This is the central character of the irreducible representation $\pi$.

## Irreducibles of the Heisenberg group

We will now quickly explain how to construct the irreducible characters of the Heisenberg group. We won't give proofs. The center is the commutator subgroup. $[G : Z(G)] = q^2$ so there are $q^2$ irreducible linear characters.

The remaining irreducible characters have nontrivial central character. Given a nontrivial linear character $\xi$ of $Z(G)$ there is a unique irreducible character $\chi_\xi$ with central character $\xi$. To obtain it, we start with the linear character $\xi$.

- We cannot extend $\xi$ to a character of $G$.
- We could induce $\xi$ to $G$ but the result would not be irreducible.

## Extend, then induce

Here what works is to first extend then induce. We have $|G| = q^3$ and $Z(G) = q$. So we look for a subgroup $A$ of order $q^2$ such that $Z(G) \subset A \subset G$. There are many such subgroups ($q+1$ to be precise) but we pick one. We could use the abelian group:

$$A = \left\{ \begin{pmatrix} 1 & & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle|\, x, y, z \in F \right\}.$$

- We extend $\xi$ to $A$. There are $q$ different such extensions.
- Then we induce the extended character from $A$ to $G$.

The result is irreducible. More remarkably, it doesn't depend on any choices we made, namely the group $A$ or the extension of $\xi$ to a character of $A$.