The affine group

Maps $F \longrightarrow F$ of the form $x \mapsto ax + b$, $a \in F^{\times}$, $b \in F$

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ 1 \end{pmatrix}$$

$$K = \left\{\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right\}: \qquad x \mapsto x + b$$

$$H = \begin{pmatrix} a & \\ & 1 \end{pmatrix}: \qquad x \mapsto ax$$

isotropy subgroup of 0.

This is a Frobenius group since if

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix}$$

fixes both x and y

$$ax + b = x, \qquad ay + b = y$$

$$a(x - y) = x - y$$

so $a = 1$, and $x + b = x$ and $b = 0$.

A useful observation

Lemma 1. *Suppose χ is the unique irreducible of degree d. Suppose θ is a* linear *character. Then if $\theta(g) \neq 1$ we must have $\chi(g) = 0$.*

Proof: $\theta\chi$ is another irreducible of degree d so $\theta\chi = \chi$ so $\theta(g)\chi(g) = \chi(g)$ so $\chi(g) = 0$.

This is helpful in many cases, D_8, Q_8, S_5 or the Frobenius group of order 20.

Reminder

Formula for the induced character ψ^G:

$$\psi^G(g) = \frac{1}{|K|}\sum_{x \in G} \dot{\psi}(xgx^{-1})$$

Or

$$\psi^G(g) = \sum_x \dot{\psi}(xgx^{-1})$$

summing over a set of coset representatives for $K\backslash G$

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in K \\ 0 & \text{otherwise.} \end{cases}$$

D_8 and Q_8

The character tables of D_8 and Q are very similar.

$$D_8 = \langle t, s | t^4 = s^2 = 1, sts^{-1} = t^{-1} \rangle$$

$$Q = \langle i, j, z | i^4 = j^4 = 1, i^2 = j^2 = z, zi = iz, zj = jz, jij^{-1} = i^{-1} \rangle.$$

Differences:

- Every subgroup of Q is normal.
- D_8 has five elements of order 2, Q only has one.

Similarities:

- center has order two, $G/Z(G) \cong Z_2 \times Z_2$
- character values are same.

	1	1	2	2	2
	1	t^2	t	s	st
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

	1	1	2	2	2
	1	z	i	j	ij
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	2	-2	0	0	0

A real valued irreducible character of any group comes from one of the two possible sources:

(1) A representation $G \longrightarrow \mathrm{GL}(V_0)$ where V_0 is a *real* vector space.

(2) A representation $G \longrightarrow \mathrm{GL}(V_0)$ where V_0 is a *quaternionic* vector space.

Both cannot be simultaneously true!

$$D_8 \longrightarrow \mathrm{GL}(2, \mathbb{R}) \longrightarrow \mathrm{GL}(2, \mathbb{C})$$

$$Q \longrightarrow \mathrm{GL}(1, \mathbb{H}) \longrightarrow \mathrm{GL}(2, \mathbb{C})$$

These two groups are *isoclinic*.