Lecture 13. Permutation Characters (II)

Daniel Bump

May 21, 2020

Semidirect Products are Ubiquitous

Today we will begin by reviewing the theory of semidirect products, which is in Dummit and Foote Section 5.5. We recall that if H and N are subgroups of G and one of them is normal, say N, then HN is a group. If both of them are normal, we can say a lot.

If we look at a lot of finite groups, many of them we encounter are semidirect products. The semidirect product construction is a general way of constructing many groups. For example we computed the character table of the group of order 21:

$$\langle a, b | a^7 = b^3 = 1, bab^{-1} = a^2 \rangle.$$

But how do we know this is a valid presentation of a group of order 21? This is easy if we know about semidirect products.

Direct Productions

Proposition

Let G be a finite group and H, N subgroups. Suppose that both H and N are normal, $H \cap N = \{1\}$ and $HN = G$. Then $G \cong H \times N$.

To prove this, the key point is to show that H and N centralize each other, that is, if $h \in H$ and $n \in N$ then $hn = nh$. Indeed, consider the commutator $[h, n] = hnh^{-1}n^{-1}$. We have

$$[h, n] = h(nhn^{-1})^{-1} = (hnh^{-1})n^{-1}.$$

The first expression shows $[h, n] \in H$ and the second shows $[h, n] \in N$. Since $H \cap N = \{1\}$, we have $[h, n] = 1$ so $hn = nh$. With this in hand, it is easy to show that $\phi : H \times N \longrightarrow G$ defined by $\phi(h, n) = hn$ is a group isomorphism.

The definition of a Semidirect Product

Now let H, N be subgroups of G as before but only assume that one of them (say N) is normal. This guarantees that $HN = \{hn | h \in H, n \in N\}$ is a group. If $HN = G$ then we call G the semidirect product and we use the notation $H \ltimes N$ or $N \rtimes H$ for this group. (The triangle in the symbol $\ltimes$ or $\rtimes$ points towards the normal subgroup.)

Examples

- The group $S_n = A_n \rtimes Z_2$ where $Z_2 = \langle (12) \rangle$.

- The dihedral group D_{2n} with the presentation

$$\langle t, s | t^n = s^2 = 1, sts^{-1} = t^{-1} \rangle$$

is the semidirect product of the normal subgroup $\langle t \rangle$ and the group $\langle s \rangle$ of order 2.

- The group A_4 is the semidirect product $Z_3 \ltimes V$ where $Z_3 = \langle (123) \rangle$ is the 3-Sylow and V is the 4-group

$$V = \{1, (12)(34), (13)(24), (14)(23)\}.$$

- Suppose that G is a nonabelian group of order pq where $p < q$. Let P and Q be the Sylow subgroups of orders p and q. The Sylow theorems imply Q is normal and $G = P \ltimes Q$.

Example: The affine group

Suppose that $F = \mathbb{F}_q$ is a finite field. Consider the matrix group

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} | a \in F^\times, b \in F \right\}.$$

This is a group of order $q(q-1)$. We may think of this as the group of affine transformations $x \mapsto ax + b$ of the field F. The group T of translations is normal:

$$T = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right\}, \qquad x \mapsto x + b.$$

The complement is

$$H = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right\} \cong F^\times.$$

This group is the semidirect product $H \ltimes T$. We will see later that this is an example of a Frobenius group.

Example: Groups of order 56

Proposition

Any nonabelian group of order $56 = 8 \cdot 7$ is a semidirect product of its Sylow subgroups.

Let P and Q be Sylow subgroups of orders 8 and 7, respectively. We will show that either P is normal or Q is normal.
Indeed, suppose Q is not normal. Then by the Sylow theorem it has 8 conjugates. Each contains 6 elements of order 7 so there are only

$$8 = 56 - 6 \times 8$$

elements of G that are not of order 7. These must comprise the 2-Sylow P which is then obviously normal. So either P or Q is normal, though we cannot say which. In either case, G is a semidirect product.

Kernel and Complement

We note that if $G = H \ltimes N$ then $H \cong G/N$. Indeed, consider the composite homomorphism

$$H \longrightarrow G \longrightarrow G/N$$

where the first map is the inclusion and the second the projection. Since $H \cap N = 1$, this map is injective, and both groups have the same order $|H|$.

However if G is a group and N a normal subgroup there may or may not be a group H such that $HN = G$ and $H \cap N = 1$. If there is such a group, we will call H the complement and N the kernel.

The reconstruction problem

Suppose we start with just the data H and N. Can we construct a group G such that $G = H \ltimes N$? The answer is yes, but there may be many such groups, for example the direct product, so we need a bit more information to reconstruct G.

First suppose that we already have G with subgroups H and N. We have a homomorphism $\theta : H \longrightarrow \mathrm{Aut}(N)$, the action by conjugation, so

$$\theta(h)n = hnh^{-1}, \qquad h \in H, n \in N.$$

Let us say that the semidirect product $G = H \ltimes N$ realizes the homomorphism $\theta : H \longrightarrow \mathrm{Aut}(N)$.

The main theorem about semidirect products

Theorem

Let H and N be groups and let $\theta : H \longrightarrow \mathrm{Aut}(N)$ be a homomorphism. Then there exists a semidirect product $G = H \ltimes N$ realizing the homomorphism θ.

To prove this, let G be the set of ordered pairs $\{(n, h) | n \in N, h \in H\}$. As a set this is the Cartesian product, but we modify the multiplication using θ:

$$(n, h)(n', h') = (n(\theta(h)n'), hh').$$

Succinctly, when h moves across n', the homomorphism $\theta(h)$ is applied.

Proof, continued

It is easy to check that this multiplication makes G into a group, with subgroups

$$\{(1,h)\} \cong H, \qquad \{(n,1)\} \cong N.$$

Thus G is a semidirect product, and it is also easy to check that it realizes the homomorphism $\theta : H \longrightarrow \mathrm{Aut}(N)$.

We say that the group G constructed this way from H, N and $\theta : H \longrightarrow \mathrm{Aut}(N)$ is the external semidirect product, and we may use the notation $H \ltimes_\theta N$ or $N \rtimes_\theta H$ to indicate this.

Review: Permutation characters

We have discussed the permutation character associated with a permutation representation; if G acts on a set X, define $\chi_X(g)$ to be the number of fixed points of g on X. We saw that $\chi^\circ(g) = \chi_X(g) - 1$ is sometimes an irreducible character of G, in fact it is irreducible if the group action is doubly transitive.

Example: S_5

Let us give another example of how this is useful. Let us compute the character table of S_5. Three characters are easy: the trivial character, the sign character and the permutation character.

	1	20	15	24	10	30	20
	1	(123)	(12)(34)	(12345)	(12)	(1234)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	1	0	-1	2	0	-1

S_5, continued

We can get another degree 4 character by tensoring $\chi_2\chi_3$, and that will still be irreducible. Call that χ_4. The remaining character degrees must sum to

$$d_5^2 + d_6^2 + d_7^2 = 120 - 1^2 - 1^2 - 4^2 - 4^2 = 86.$$

There are three ways of writing 86 as a sum of 3 squares:

$$86 = 1^2 + 2^2 + 9^2 = 1^2 + 6^2 + 7^2 = 5^2 + 5^2 + 6^2$$

but we know that S_5 has exactly two linear characters and we've already taken those into account. Therefore S_5 has two more representations of degree 5 and one of degree 6.

The action on 5-Sylows

One degree 5 representation can be obtained by noting that S_5 has six 5-Sylow subgroups:

$$\begin{array}{lclcll}
A & = & \langle(12345)\rangle & \Rightarrow & \langle(23145)\rangle = \langle(12435)\rangle & C \\
B & = & \langle(12354)\rangle & \Rightarrow & \langle(23154)\rangle = \langle(12534)\rangle & E \\
C & = & \langle(12435)\rangle & \Rightarrow & \langle(23415)\rangle = \langle(12453)\rangle & D \\
D & = & \langle(12453)\rangle & \Rightarrow & \langle(23451)\rangle = \langle(12345)\rangle & A \\
E & = & \langle(12534)\rangle & \Rightarrow & \langle(23514)\rangle = \langle(12543)\rangle & F \\
F & = & \langle(12543)\rangle & \Rightarrow & \langle(23541)\rangle = \langle(12354)\rangle & B
\end{array}$$

So S_5 acts on the set $Y = \{A, B, C, D, E, F\}$ by conjugation. For example

$$(123)A(123)^{-1} = \langle(123)(12345)(123)^{-1}\rangle = \langle(23145)\rangle = \langle(12435)\rangle = C$$

and conjugation by (123) produces $(ACD)(BEF)$. There are no fixed points, so $\chi_Y((123)) = 0$ and $\chi_Y^\circ((123)) = -1$

The reduced character

Here are the effects of all of the conjugacy classes

		χ_Y°
1	1	5
(123)	$(ACD)(BEF)$	-1
$(12)(34)$	$(AE)(CF)$	1
(12345)	$(BDFEC)$	0
(12)	$(AF)(BD)(CE)$	-1
(1234)	$(ADEB)$	1
$(123)(45)$	$(AEDBCF)$	-1

Let us call the character $\chi_Y^\circ = \chi_5$. We get another irreducible character $\chi_6 = \chi_2\chi_5$.

The character table, nearly finished

We now have this much of the character table:

	1	20	15	24	10	30	20
	1	(123)	(12)(34)	(12345)	(12)	(1234)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	1	0	-1	2	0	-1
χ_4	4	1	0	-1	-2	0	1
χ_5	5	-1	1	0	-1	1	-1
χ_6	5	-1	1	0	1	-1	1
χ_7	6						

We will leave the degree 6 character for another time.

Equivalence of Group Actions

Let G act on two sets X and Y. The two actions are called equivalent if there is a bijection $f : X \longrightarrow Y$ such that $f(gx) = gf(x)$.

Let H be a subgroup of G, not necessarily normal. Let G/H be the set of left cosets xH. (It is the quotient group of $H \lhd G$, otherwise it is just a set.) We have an action of G on left cosets by $g(xH) = (gx)H$

Let G act on X. If $x \in X$, let $G_x = \{g \in G | gx = x\}$. This is the stabilizer or isotropy subgroup. If the action on X is transitive, the isotropy subgroups are all conjugate. So (assuming transitivity) we pick an $x \in G$ and let $H = G_x$ be the isotropy subgroup.

The orbit-stabilizer theorem

Proposition (The Orbit-Stabilizer theorem)

Let G act transitively on X and let $x \in X$. Then the action of G on X is equivalent to the action on G/H.

Although the proof of this is easy, this fact is fundamental and should be emphasized more in Dummit and Foote, Chapter 4.

To prove this, define a map $\phi : G/H \longrightarrow X$ by $\phi(gH) = gx$. Note that

$$gx = g'x \quad \Leftrightarrow \quad g^{-1}g'x = x \quad \Leftrightarrow \quad g^{-1}g' \in H \quad \Leftrightarrow \quad gH = g'H.$$

This shows that ϕ is well-defined and injective. It is surjective since G acts transitively on X. It is easy to check that this bijection is an equivalence of group actions.

Permutation representations as induced representations

Proposition

Let G act transitively on X, and let $H = G_x$ be an isotropy subgroup. Then the permutation character χ_X is the character induced from the trivial representation of H.

We will deduce this from the formula for the character of the induced representation. If ψ is a character of H

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

We showed that the induced character

$$\psi^G(g) = \sum_{Hk \in H\backslash G} \dot{\psi}(kgk^{-1}).$$

This is the way we wrote the formula on Thursday, but today we want to sum over left cosets, so we replace k by k^{-1} and

$$\psi^G(g) = \sum_{kH \in G/H} \dot{\psi}(k^{-1}gk).$$

Take ψ to be the trival character of the stabilizer H. We will show that this is the number $\chi(g)$ of fixed points of G. By the orbit stabilizer theorem, we may identify $X = G/K$ and the sum

$$\sum_{kH \in G/H} \dot{\psi}(k^{-1}gk)$$

counts the cosets kH such that $k^{-1}gk \in H$, that is, such that $gkH = kH$. So $\psi^G(g)$ is the number of fixed points, $\chi_X(g)$.

Problem 4d in Section 19.1

Now let us see how the Orbit Stabilizer Theorem could be used to reduce the work to do one of the homework problems, Problem 4d in Section 19.1. This problem asks us to compute the character of the permutation action of $G = S_5$ on the cosets of the group

$$H = \langle (123), (12), (45) \rangle \cong S_3 \times S_2.$$

Let us call this character θ.

One way: This group of order 12 is the normalizer of the 3-Sylow subgroup $\langle (123) \rangle$. Since H has index 10, there are 10 Sylow subgroups. So to compute $\theta(g)$ we could count the number of fixed points in this action. So $\theta(g)$ is the number of 3-Sylow subgroups normalized g. This would be similar to the way we computed χ_5.

A simple way

But instead let us note that G acts (transitively) on the 10 two-element subsets of $\{1, 2, 3, 4, 5\}$, and H is the stabilizer of the subset $\{4, 5\}$. This means that we can just compute the number of 2-element subsets fixed by g.

For example, (123) fixes the 2-element subset $\{4, 5\}$ and no others. Thus $\theta((123)) = 1$. The cycle (12345) doesn't fix any two element sets. But $(12)(34)$ fixes the two element subsets $\{1, 2\}$ and $\{3, 4\}$ so $\theta((12)(34)) = 2$. The transposition (12) fixes $\{1, 2\}$, $\{3, 4\}$, $\{3, 5\}$ and $\{4, 5\}$ so $\theta((12)) = 4$.

The character θ

	1	20	15	24	10	30	20
	1	(123)	(12)(34)	(12345)	(12)	(1234)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	1	0	-1	2	0	-1
χ_4	4	1	0	-1	-2	0	1
χ_5	5	-1	1	0	-1	1	-1
χ_6	5	-1	1	0	1	-1	1
χ_7	6						
θ	10	1	2	0	4	0	1

Conclusion

Now we compute

$$\langle \theta, \theta \rangle = \frac{1}{120}(10^2 + 20 \times 1^2 + 15 \times 2^2 + 24 \times 0^2 + 10 \times 4^2 + 20 \times 1^2) = 3$$

so we expect θ to be the sum of three irreducible characters, each with multiplicity one. I will leave the problem here.