Lecture 12. Induced Characters

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The idea of an induced representation

If G is a finite group and H a subgroup, there is a functor from $\mathbb{C}[H]$-modules to $\mathbb{C}[G]$-modules, or equivalently from representations of H to representations of G, called induction.

Today we will take a purely character-theoretic approach to this. This ad hoc approach, based on a couple of formulas, doesn't give any real insight into the representations, but it will give us quick access to the character of the induced representation, and we will find applications.

Frobenius reciprocity

We will make use of the characters of both G and its subgroup H. We will denote by $\langle\ ,\ \rangle_G$ and $\langle\ ,\ \rangle_H$ the L^2 inner products on both these groups. Today we will prove:

Theorem (Frobenius reciprocity)

Let H be a subgroup of G. Let χ be a character of H. Then there is a character χ^G of G such that if ψ is another character of G then

$$\langle\chi^G, \psi\rangle_G = \langle\chi, \psi\rangle_H.$$

Note that on the right-hand side, we are restricting the character ψ of G to the subgroup H to compute the inner product.

Reduction of the problem

Let $\psi_1, \cdots, \psi_k$ be the distinct irreducible representations of G. Assuming we have constructed a candidate for the induced character χ^G, a necessary and sufficient condition for

$$\langle \chi^G, \psi \rangle_G = \langle \chi, \psi \rangle_H$$

to be true for all ψ is that it is true for the ψ_i. Indeed, the ψ_i span the space of class functions, and both sides are linear in ψ.

To prove that χ^G exists, we will give an ad hoc definition that makes Frobenius reciprocity true for the ψ_i.

Definition of χ^G

Let us define

$$\chi^G = \sum_{i=1}^{k} m_i \psi_i, \qquad m_i = \langle \chi, \psi_i \rangle_H.$$

Note that the m_i are nonnegative integers so this is a character. Now to check Frobenius reciprocity for $\psi = \psi_j$ we compute

$$\langle \chi^G, \psi_j \rangle_G = \sum_{i=1}^{k} m_i \langle \psi_i, \psi_j \rangle = m_j = \langle \chi, \psi_j \rangle_H.$$

Thus we have proved Frobenius reciprocity for this χ^G.

A formula for χ^G

We now know that there exists a character χ^G. In order to work with it it will be useful to have a formula for it. To define this, note that χ is defined on H, not all of G. So we define

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

We have

$$\chi^G(x) = \frac{1}{|H|} \sum_{g \in G} \dot{\chi}(gxg^{-1}).$$

Proof

Denote the right hand side as $\tilde{\chi}(g)$. Obviously this is a class function on G. So to show that it equals χ^G, we need only show that both have the same inner product with characters of G. By Frobenius reciprocity, this means that we need to prove

$$\langle \tilde{\chi}, \psi \rangle_G = \langle \chi, \psi \rangle_H.$$

So we calculate:

$$\langle \tilde{\chi}, \psi \rangle = \frac{1}{|G|} \sum_{x \in G} \tilde{\chi}(x) \overline{\psi(x)} = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|H|} \sum_{g \in G} \dot{\chi}(gxg^{-1}) \overline{\psi(x)}.$$

Because ψ is a class function, we may replace $\psi(x)$ with $\psi(gxg^{-1})$. Then we interchange the order of summation to obtain

$$\langle \tilde{\chi}, \psi \rangle = \frac{1}{|H|} \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \dot{\chi}(gxg^{-1}) \overline{\psi(gxg^{-1})}.$$

Proof, concluded

Now with g fixed, we make the substitution $x \to g^{-1}xg$ and this becomes

$$\frac{1}{|H|}\frac{1}{|G|}\sum_{x\in G}\sum_{g\in G}\dot{\chi}(x)\overline{\psi(x)} = \frac{1}{|H|}\sum_{x\in G}\dot{\chi}(x)\overline{\psi(x)}.$$

Now since

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

we may restrict the summation to H and we have proved

$$\langle \tilde{\chi}, \psi \rangle_G = \langle \chi, \psi \rangle_H.$$

As noted, this implies the advertised formula for the induced character.

Summing over coset representatives

There is a simplification of the formula:

$$\chi^G(x) = \frac{1}{|H|}\sum_{g\in G}\dot{\chi}(gxg^{-1}).$$

Although $\dot{\chi}$ is not a class function for G, it is still true

$$\dot{\chi}(hxh^{-1}) = \dot{\chi}(x).$$

This means that we can group the elements g into cosets Hg. We will denote by $H\backslash G$ the set of such cosets. The summand $\dot{\chi}(gxg^{-1})$ is constant on these cosets. So if we take just one representative from each coset we may drop the factor $|H|$ in the denominator. We obtain the formula

$$\boxed{\chi^G(x) = \sum_{Hg\in H\backslash G}\dot{\chi}(gxg^{-1}).}$$

The degree of the induced representation

The induced character is of course the character of a representation, and we will study the representation directly by other methods. But the character does contain complete information about the representation, since the character determines the representation

Proposition

Let d be the degree (dimension) of the represention is $d[G:H]$.

To prove this, we note that the degree d of the representation with character χ is $d = \chi(1)$. So the degree $\chi^G(1)$ of the induced representation is

$$\sum_{Hg \in H\backslash G} \dot{\chi}(g \cdot 1 \cdot g^{-1}) = |H\backslash G| \chi(1) = [G:H]d.$$

Using induced representations to construct irreducibles

If χ is an irreducible character, the induced character χ^G may or may not be irreducible.

It may be proved that if G is a p-group then every irreducible character is induced from a linear character. For more general groups, Brauer proved that for any finite group, characters induced from linear characters generate the group of virtual characters, and this leads to a strategy for constructing all irreducible characters.

To summarize, linear characters are a easy to determine, so a good strategy for constructing representations of the group is to take characters induced from linear characters of various subgroups; this may produce irreducibles, and if it doesn't, the irreducibles may then be found by taking linear combinations.

Example: the dihedral group D_8

Let us do an example. We consider the dihedral group

$$D_8 = \langle t, s | t^4 = s^2 = 1, sts^{-1} = t^{-1} \rangle.$$

The commutator subgroup is the center $\langle t^2 \rangle$ of order 2 and the quotient $D_8/\langle t^2 \rangle \cong Z_2 \times Z_2$. So we obtain four linear characters.

Conj. Class (size)	$1(1)$	$t(2)$	$t^2(1)$	$s(2)$	$st(2)$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1

The dihedral group, continued

There is also a two dimensional irreducible character. Can we construct this two dimensional character as an induced character?

Remembering that induction from H multiplies the degree by $[G : H]$, we are looking for a subgroup of index two. An obvious candidate is the cyclic subgroup $\langle t \rangle$. Let us induce the linear character of this cyclic group of order 4 such that $\chi(t) = i$.

The dihedral group, continued

To use the formula

$$\chi^G(x) = \sum_{Hg \in H\backslash G} \dot{\chi}(gxg^{-1}),$$

we need coset representatives for $H\backslash G$. We take $\{1, s\}$ and the formula

$$\chi^G(x) = \dot{\chi}(x) + \dot{\chi}(sxs^{-1}).$$