

$$\begin{array}{ccccc}
& & \text{induced} & & \\
G & \chi^G & \psi & \langle \chi^G, \psi \rangle_G & \\
& \uparrow & \downarrow & \parallel & \\
H & \chi & \psi|_H & \langle \chi, \psi|_H \rangle_H & \\
& & \text{restricted} & &
\end{array}$$

Finishing the D_8 example

$$G = \langle t, s \mid t^4 = s^2 = 1, sts^{-1} = t^{-1} \rangle$$

$$H = \langle t \rangle$$

$$\chi: \begin{array}{|c|c|c|c|} \hline 1 & t & t^2 & t^3 \\ \hline 1 & i & -1 & -i \\ \hline \end{array}$$

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G(x) = \sum_{Hg \in H \backslash G} \dot{\chi}(gxg^{-1})$$

$$\dot{\chi}: \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline x & 1 & t & t^2 & t^3 & s & st & st^2 & st^3 \\ \hline \dot{\chi}(x) & 1 & i & -1 & -i & 0 & 0 & 0 & 0 \\ \hline \dot{\chi}(sxs^{-1}) & 1 & -i & -1 & i & 0 & 0 & 0 & 0 \\ \hline \chi^G(x) & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

Take as our coset representatives $\{1, s\}$

	1	t	t ²	s	st
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

Dihedral group D_{10}

$$G = \langle t, s \mid t^5 = s^2 = 1, sts^{-1} = t^{-1} \rangle$$

	1	2	2	5
	1	t	t ²	s
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	α	β	0
χ_4	2	β	α	0

$$\sum d_i^2 = 10$$

To construct a character of degree 2, induce from subgroup of index 2.

x	1	t	t^2	t^3	t^4	s	st	st^2	st^3	st^4
$\chi(x)$	1	ζ	ζ^2	ζ^3	ζ^4	0	0	0	0	0
$\dot{\chi}(sxs^{-1})$	1	ζ^{-1}	ζ^{-2}	ζ^{-3}	ζ^{-4}	0	0	0	0	0
$\chi_3 = \chi^G$	2	$\zeta + \zeta^{-1}$	$\zeta^2 + \zeta^{-2}$	$\zeta^2 + \zeta^{-2}$	$\zeta + \zeta^{-1}$	0	0	0	0	0

$$\zeta = e^{2\pi i/5}, \quad \alpha = \zeta + \zeta^{-1} = 2\cos\left(\frac{2\pi}{5}\right), \quad \beta = \zeta^2 + \zeta^{-2} = 2\cos\left(\frac{4\pi}{5}\right)$$

Coset representatives for $H \backslash G$: $\{1, s\}$

Nonabelian group of order 21

Suppose $|G| = pq$ where p, q distinct primes. The number of p -Sylows is either 1 or q and if it is q , it must be $\equiv 1 \pmod p$ (Sylow theorem) so this can happen only if $q \equiv 1 \pmod p$. Suppose $|G| = 21$. Since $7 \equiv 1 \pmod 3$ we can imagine there is a nonabelian group with 7 three-Sylows. But the subgroup of order 7 must be normal.

$$\langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^k \rangle$$

If we conjugate 3 times $a = b^3 a b^{-3} = a^{k^3}$ so we need $k^3 \equiv 1 \pmod 7$. $k = 2$ will work so hopefully

$$\langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2 \rangle$$

defines a nonabelian group of order 21. (It does.)

The commutator subgroup G' is $\langle a \rangle$. (It is clear that $\langle a \rangle$ is normal and $G/\langle a \rangle$ is abelian so $\langle a \rangle \supset G'$.) $G/G' \cong Z_3$ and so we get three linear characters:

$$\{1\}, \{a, a^2, a^4\}, \{a^3, a^5, a^6\}, \{ba^k\}, \{b^2 a^k\}$$

	1	a	a^2	b	b^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ
χ_4	3	γ	δ	0	0
χ_5	3	δ	γ	0	0

$$d_4^2 + d_5^2 = 21 - 1^2 - 1^2 - 1^2 = 18$$

$$\rho = e^{2\pi i/3}, \quad \zeta = e^{2\pi i/7}$$

Find coset representatives for $H \backslash G$. Take $\{1, b, b^2\}$

$$bab^{-2} = a^2, \quad b^2ab^{-2} = a^4$$

x	1	a	a^2	a^3	a^4	a^5	a^6	b	...	b^2	...
$\dot{\chi}(x)$	1	ζ	ζ^2	ζ^3	ζ^4	ζ^5	ζ^6	0		0	
$\dot{\chi}(bxb^{-1})$	1	ζ^2	ζ^4	ζ^6	ζ	ζ^3	ζ^5				
$\dot{\chi}(b^2xb^{-2})$	1	ζ^4	ζ	ζ^5							
$\chi^G(x)$	3	γ	γ	δ	γ	δ	δ	0		0	

$$\gamma = \zeta + \zeta^2 + \zeta^4, \quad \delta = \zeta^3 + \zeta^5 + \zeta^6$$

$$\frac{1}{21}(3^2 + 3\gamma\bar{\gamma} + 3\delta\bar{\delta}) = 1$$